3 Application of the Governing Equations to Turbulent Flow

To quantitatively describe and forecast the state of the boundary layer, we turn to the equations of fluid mechanics that describe the dynamics and thermodynamics of the gases in our atmosphere. Motions in the boundary layer are slow enough compared to the speed of light that the Galilean/Newtonian paradigm of classical physics applies. These equations, collectively known as the ***equations of motion***, contain time and space derivatives that require initial and boundary conditions for their solution.

Although the equations of motion together with other conservation equations can be applied directly to turbulent flows, rarely do we have sufficient initial and boundary condition information to resolve all turbulent scales down to the smallest eddy. We often don't even care to forecast all eddy motions. For simplicity, we instead pick some cut-off eddy size below which we include only the statistical effects of turbulence. In some mesoscale and synoptic models the cutoff is on the order of 10 to 100 km, while for some boundary layer models known as ***large eddy simulation models*** the cutoff is on the order of 100 m.

The complete set of equations as applied to the boundary layer are so complex that no analytical solution is known. As in other branches of meteorology, we are forced to find approximate solutions. We do this by either finding exact analytical solutions to simplified subsets of the equations, or by finding approximate numerical solutions to a more complete set of equations. Both approximations are frequently combined to allow boundary layer meteorologists to study particular phenomena.

In this chapter we start with the basic governing equations and statistically average over the smaller eddy sizes. Along the way we demonstrate simplifications based on boundary layer scaling arguments. Numerical methods for solving the resulting set of equations are not covered.

3.1 Methodology

Because the upcoming derivations are sometimes long and involved, it is easy "to lose sight of the forest for the trees". The following summary gives the steps that will be taken in the succeeding sections to develop prognostic equations for mean quantities such as temperature and wind:

Step 1. Identify the basic governing equations that apply to the boundary layer.
Step 2. Expand the total derivatives into the local and advective contributions.
Step 3. Expand dependent variables within those equations into mean and turbulent (perturbation) parts.
Step 4. Apply Reynolds averaging to get the equations for mean variables within a turbulent flow.
Step 5. Add the continuity equation to put the result into flux form.

Additional steps take us further towards understanding the nature of turbulence itself:

Step 6. Subtract the equations of step 5 from the corresponding ones of step 3 to get equations for the turbulent departures from the mean.
Step 7. Multiply the results of step 6 by other turbulent quantities and Reynolds average to yield prognostic equations for turbulence statistics such as kinematic flux or turbulence kinetic energy.

Section 3.2 covers steps 1 and 2. Section 3.3 takes a side road to look at some simplifications and scaling arguments. In section 3.4 we get back on track and utilize steps 3-5 to derive the desired prognostic equations. After a few more simplifications in section 3.5, a summary of the governing equations for mean variables in turbulent flow is presented.

Steps 6 and 7 are addressed in Chapters 4 and 5.

3.2 Basic Governing Equations

Five equations form the foundation of boundary layer meteorology: the equation of state, and the conservation equations for mass, momentum, moisture, and heat. Additional equations for scalar quantities such as pollutant concentration may be added. It is assumed that the reader has already been exposed to these equations; hence, the derivations are not given here.

3.2.1 Equation of State (Ideal Gas Law)

The ideal gas law adequately describes the state of gases in the boundary layer:

$$p = \rho_{air} \, \Re \, T_v \qquad (3.2.1)$$

where p is pressure, ρ_{air} is the density of moist air, T_v is the virtual absolute temperature, and $\Re$ is the gas constant for **dry** air ($\Re$ = 287 J·K^{-1} kg^{-1}). Sometimes, the density of moist air is abbreviated as ρ for simplicity.

3.2.2 Conservation of Mass (Continuity Equation)

Two equivalent forms of the continuity equation are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_j)}{\partial x_j} = 0 \tag{3.2.2a}$$

and

$$\frac{d\rho}{dt} + \rho \frac{\partial U_j}{\partial x_j} = 0 \tag{3.2.2b}$$

where the definition of the total derivative is used to convert between these forms.

If $\mathbb{V}$ and $\mathbb{L}$ are typical velocity and length scales for the boundary layer, then it can be shown (Businger, 1982) that $(d\rho/dt)/\rho \ll \partial U_j/\partial x_j$ if the following conditions are met: (1) $\mathbb{V} \ll 100$ m/s; (2) $\mathbb{L} \ll 12$ km; (3) $\mathbb{L} \ll C_s^2/g$; and (4) $\mathbb{L} \ll C_s/f$, where C_s is the speed of sound and f is frequency of any pressure waves that might occur. Since these conditions are generally met for all turbulent motions smaller than mesoscale, (3.2.2b) reduces to

$$\frac{\partial U_j}{\partial x_j} = 0 \tag{3.2.2c}$$

This is the ***incompressibility*** approximation.

3.2.3 Conservation of Momentum (Newton's Second Law)

As presented at the end of section 2.8.2, one form for the momentum equation is

$$\underset{\text{I}}{\frac{\partial U_i}{\partial t}} + \underset{\text{II}}{U_j \frac{\partial U_i}{\partial x_j}} = \underset{\text{III}}{- \delta_{i3}\, g} \underset{\text{IV}}{- 2\, \varepsilon_{ijk}\, \Omega_j\, U_k} \underset{\text{V}}{- \frac{1}{\rho}\frac{\partial p}{\partial x_i}} + \underset{\text{VI}}{\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}} \tag{3.2.3a}$$

Term I represents storage of momentum (inertia).
Term II describes advection.
Term III allows gravity to act vertically.
Term IV describes the influence of the earth's rotation (Coriolis effects).
Term V describes pressure-gradient forces.
Term VI represents the influence of viscous stress.

In term IV, the components of the angular velocity vector of the earth's rotation Ω_j are $[0, \omega \cos(\phi), \omega \sin(\phi)]$ where ϕ is latitude and $\omega = 2\pi$ radians/24h $= (7.27\text{x}10^{-5}\ s^{-1})$ is the angular velocity of the earth. Often term IV is written as $+ f_c\, \varepsilon_{ij3}\, U_j$, where the ***Coriolis parameter*** is defined as $f_c = 2\,\omega \sin\phi = (1.45\text{x}10^{-4}\ s^{-1}) \sin\phi$. For a latitude of about 44° (e.g., southern Wisconsin), $f_c = 10^{-4}\ s^{-1}$.

To a close approximation, air in the atmosphere behaves like a Newtonian fluid. Thus, the expression for viscous stress from section 2.9.3 allows us to write term VI as:

$$\text{Term VI} = \left(\frac{1}{\rho}\right) \frac{\partial}{\partial x_j} \left\{ \mu \left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] - \left(\frac{2}{3}\right) \mu \left[\frac{\partial U_k}{\partial x_k} \right] \delta_{ij} \right\}$$

where the bulk viscosity coefficient μ_B was assumed to be near zero. Upon applying the derivative to each term, assuming that the viscosity μ is not a function of position, and rearranging, this expression can be written as:

$$\text{Term VI} = \left(\frac{\mu}{\rho}\right) \left\{ \frac{\partial^2 U_i}{\partial x_j^2} + \frac{\partial}{\partial x_i} \left[\frac{\partial U_j}{\partial x_j} \right] - \left(\frac{2}{3}\right) \frac{\partial}{\partial x_i} \left[\frac{\partial U_k}{\partial x_k} \right] \right\}$$

By assuming incompressibility, this reduces to

$$\text{Term VI} = \nu \frac{\partial^2 U_i}{\partial x_j^2}$$

where the kinematic viscosity, ν, has been substituted for μ/ρ.

Substituting this back into (3.2.3a) gives the form for the momentum equation that is most often used as a starting point for turbulence derivations:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3}\, g + f_c\, \varepsilon_{ij3}\, U_j - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} \qquad (3.2.3b)$$

$$\text{I} \qquad \text{II} \qquad \text{III} \qquad \text{IV} \qquad \text{V} \qquad \text{VI}$$

where each term represents the same process as before.

3.2.4 Conservation of Moisture

Let q_T be the total specific humidity of air; namely, the mass of water (all phases) per unit mass of moist air. The conservation of water substance can be written, assuming incompressibility, as

$$\underset{\text{I}}{\frac{\partial q_T}{\partial t}} + \underset{\text{II}}{U_j \frac{\partial q_T}{\partial x_j}} = \underset{\text{VI}}{\nu_q \frac{\partial^2 q}{\partial x_j^2}} + \underset{\text{VII}}{\frac{S_{q_T}}{\rho_{air}}} \qquad (3.2.4a)$$

where ν_q is the molecular diffusivity for water vapor in the air. S_{q_T} is a net moisture source term (sources - sinks) for the remaining processes not already included in the equation. Its units are: mass of total water per unit volume per unit time.

By splitting the total humidity into vapor (q) and non-vapor (q_L) parts using $q_T = q + q_L$ and $S_{q_T} = S_q + S_{q_L}$, (3.2.4a) can be rewritten as a pair of coupled equations

$$\frac{\partial q}{\partial t} + U_j \frac{\partial q}{\partial x_j} = \nu_q \frac{\partial^2 q}{\partial x_j^2} + \frac{S_q}{\rho_{air}} + \frac{E}{\rho_{air}} \qquad (3.2.4b)$$

and

$$\underset{\text{I}}{\frac{\partial q_L}{\partial t}} + \underset{\text{II}}{U_j \frac{\partial q_L}{\partial x_j}} = \underset{\text{VI}}{} + \underset{\text{VII}}{\frac{S_{q_L}}{\rho_{air}}} - \underset{\text{VIII}}{\frac{E}{\rho_{air}}} \qquad (3.2.4c)$$

where E represents the mass of water vapor per unit volume per unit time being created by a phase change from liquid or solid. The convergence of falling liquid or solid water (e.g., precipitation) that is not advecting with the wind is included as part of term VII. It has been assumed in (3.2.4c) that molecular diffusion has a negligible effect on liquid and solid precipitation or cloud particles.

Terms I, II, and VI are analogous to the corresponding terms in the momentum equation. Term VII is a net body source term, and term VIII represents the conversion of solid or liquid into vapor.

3.2.5 Conservation of Heat (First Law of Thermodynamics)

The First Law of Thermodynamics describes the conservation of enthalpy, which includes contributions from both sensible and latent heat transport. In other words, the water vapor in air not only transports sensible heat associated with its temperature, but it has the potential to release or absorb additional latent heat during any phase changes that might occur. To simplify the equations describing enthalpy conservation, micrometeorologists often utilize the phase change information, E, contained in the moisture conservation equations. Thus, an equation for θ can be written

$$\frac{\partial \theta}{\partial t} + U_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho C_p}\left(\frac{\partial Q_j^*}{\partial x_j}\right) - \frac{L_p E}{\rho C_p} \qquad (3.2.5)$$

I II VI VII VIII

where ν_θ is the thermal diffusivity, and L_p is the latent heat associated with the phase change of E. The values for latent heat at 0°C are $L_v = 2.50 \times 10^6$ J/kg (gas:liquid), $L_f = 3.34 \times 10^5$ J/kg (liquid:solid), and $L_s = 2.83 \times 10^6$ J/kg of water (gas:solid).

Q^*_j is the component of net radiation in the j^{th} direction. The specific heat for ***moist*** air at constant pressure, C_p, is approximately related to the specific heat for dry air, $C_{pd} = 1004.67$ J kg^{-1} K^{-1}, by $C_p = C_{pd}(1 + 0.84\,q)$. Given typical magnitudes of q in the boundary layer, it is important not to neglect the moisture contribution to C_p.

Terms I, II, and VI are the storage, advection, and molecular diffusion terms, as before. Term VII is the "body source" term associated with radiation divergence. Term VIII is also a "body source" term associated with latent heat released during phase changes. These body source terms affect the whole volume, not just the boundaries.

3.2.6 Conservation of a Scalar Quantity

Let C be the concentration (mass per volume) of a scalar such as a tracer in the atmosphere. The conservation of tracer mass requires that

$$\frac{\partial C}{\partial t} + U_j \frac{\partial C}{\partial x_j} = \nu_C \frac{\partial^2 C}{\partial x_j^2} + S_C \qquad (3.2.6)$$

I II VI VII

where ν_C is the molecular diffusivity of constituent C. S_C is the body source term for the remaining processes not already in the equation, such as chemical reactions. The physical interpretation of each term is analogous to that of (3.2.4c).

3.3 Simplifications, Approximations, and Scaling Arguments

Under certain conditions the magnitudes of some of the terms in the governing equations become smaller than the other terms and can be neglected. For these situations the equations become simpler — a fact that has allowed advances to be made in atmospheric dynamics that would otherwise have been more difficult or impossible.

One simplification is called the ***shallow motion approximation*** (Mahrt, 1986). This approximation is valid if all of the following conditions are true:

1) the vertical depth scale of density variations in the boundary layer is much shallower than the scale depth of the lower atmosphere. (This latter scale depth $= \rho\,(\partial\rho/\partial z)^{-1} \cong 8$ km.);
2) advection and divergence of mass at a fixed point approximately balance, leaving only slow or zero variations of density with time.
3) the perturbation magnitudes of density, temperature, and pressure are much less than their respective mean values; and

A more stringent simplification, called the ***shallow convection approximation***, requires all of the conditions above plus:

4) the mean lapse rate $(\partial T/\partial z)$ can be negative, zero, or even slightly positive. For the statically stable positive case, $(\partial T/\partial z) \ll g/\Re$, where $g/\Re = 0.0345$ K/m; and
5) the magnitude of the vertical perturbation pressure gradient term must be of the same order or less than the magnitude of the buoyancy term in the equation of motion.

This latter condition says that vertical motion is limited by buoyancy, which is origin of the term "shallow convection".

We have already employed conditions (1) and (2) to yield the incompressible form of the continuity equation. The other conditions will be applied below to yield further simplifications.

3.3.1 Equation of State

Start with the equation of state (3.2.1) and split the variables into mean and turbulent parts: $\rho = \overline{\rho} + \rho'$, $T_v = \overline{T_v} + T_v'$, $p = \overline{P} + p'$. The result can be rearranged to be

$$\frac{\overline{P}}{\Re} + \frac{p'}{\Re} = (\overline{\rho} + \rho') \cdot (\overline{T_v} + T_v')$$

or

$$\frac{\overline{P}}{\Re} + \frac{p'}{\Re} = \overline{\rho} \cdot \overline{T_v} + \rho'\,\overline{T_v} + \overline{\rho}\,T_v' + \rho'\,T_v' \qquad (3.3.1a)$$

Upon Reynolds averaging, we are left with

$$\frac{\overline{P}}{\Re} = \overline{\rho}\,\overline{T_v} + \overline{\rho' T_v'}$$

The last term is usually much smaller in magnitude than the others, allowing us to neglect it. As a result, the equation of state holds in the mean:

$$\frac{\overline{P}}{\Re} = \overline{\rho}\, \overline{T_v} \qquad (3.3.1b)$$

This is a reasonable approximation because the equation of state was originally formulated from measurements made with crude, slow-response sensors that were essentially measuring mean quantities. As we shall see in section 3.4, however, we can't make similar assumptions for the other governing equations.

Subtracting (3.3.1b) from (3.3.1a) leaves

$$\frac{p'}{\Re} = \rho'\, \overline{T_v} + \overline{\rho}\, T_v' + \rho' T_v'$$

Finally, dividing by (3.3.1b) gives

$$\frac{p'}{\overline{P}} = \frac{\rho'}{\overline{\rho}} + \frac{T_v'}{\overline{T_v}} + \frac{\rho' T_v'}{\overline{\rho}\, \overline{T_v}}$$

Using condition (3) above and the data below, one can show that the last term is smaller than the others, leaving the ***linearized perturbation ideal gas law***:

$$\frac{p'}{\overline{P}} = \frac{\rho'}{\overline{\rho}} + \frac{T_v'}{\overline{T_v}} \qquad (3.3.1c)$$

Static pressure fluctuations are associated with variations in the mass of air from column to column in the atmosphere. For the larger eddies and thermals in the boundary layer, these fluctuations may be as large as 0.01 kPa (0.1 mb), while for smaller eddies the effect is smaller. Dynamic pressure fluctuations associated with wind speeds of up to about 10 m/s also cause fluctuations of about 0.01 kPa. Thus, for most boundary layer situations, $p'/\overline{P} = 0.01 \text{ kPa} / 100 \text{ kPa} = 10^{-4}$, which is smaller than $T_v'/\overline{T_v} = 1 \text{ K} / 300 \text{ K} = 3.33 \times 10^{-3}$. For these cases we can make the shallow convection approximation [conditions (4) & (5)] to neglect the pressure term, yielding:

$$\frac{\rho'}{\overline{\rho}} = -\frac{T_v'}{\overline{T_v}} \qquad (3.3.1d)$$

Using Poisson's relationship with the same scaling as above yields:

$$\frac{\rho'}{\bar{\rho}} = -\frac{\theta_v'}{\bar{\theta}_v} \tag{3.3.1e}$$

Physically, (3.3.1e) states that air that is warmer than average is less dense than average. Although not a surprising conclusion, these equations allow us to substitute temperature fluctuations, easily measurable quantities, in place of density fluctuations, which are not so easily measured.

3.3.2 Flux Form of Advection Terms

All of the conservation equations of section 3.2 include an advection term of the form

$$\text{Advection Term} = U_j\, \partial\xi/\partial x_j$$

where ξ denotes any variable, such as a wind component or humidity. If we multiply the continuity equation (3.2.2c) by ξ, we get $\xi\, \partial U_j/\partial x_j = 0$. Since this term is equal to zero, adding it to the advection term will cause no change (other than the mathematical form). Performing this addition gives

$$\text{Advection Term} = U_j\, \partial\xi/\partial x_j + \xi\, \partial U_j/\partial x_j$$

By using the product rule of calculus, we can combine these two terms to give

$$\text{Advection Term} = \partial(\xi U_j)/\partial x_j \tag{3.3.2}$$

This is called the ***flux form*** of the advection term, because as was demonstrated in section 2.6 the product of $(\xi\, U_j)$ is nothing more than a kinematic flux.

3.3.3 Conservation of Momentum

Vertical Component. By setting $i = 3$ in (3.2.3b), we can focus on just the vertical component of momentum to study the role of gravity, density, and pressure on turbulent motions. Utilizing $U_3 = W$ and the definition of the total derivative, $dU_i/dt = \partial U_i/\partial t + U_j\, \partial U_i/\partial x_j$, gives

$$\frac{dW}{dt} = -g - \frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right) + \nu \frac{\partial^2 W}{\partial x_j^2}$$

In the following development, we will treat viscosity as a constant. Multiply the above equation by ρ and let $\rho = \overline{\rho} + \rho'$, $W = \overline{W} + w'$ and $p = \overline{P} + p'$:

$$(\overline{\rho} + \rho')\frac{d(\overline{W}+w')}{dt} = -(\overline{\rho} + \rho')\, g - \frac{\partial(\overline{P}+p')}{\partial z} + \mu \frac{\partial^2(\overline{W}+w')}{\partial x_j^2}$$

Dividing by $\overline{\rho}$ and rearranging gives:

$$\left(1 + \frac{\rho'}{\overline{\rho}}\right)\frac{d(\overline{W}+w')}{dt} = -\frac{\rho'}{\overline{\rho}} g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \nu \frac{\partial^2(\overline{W}+w')}{\partial x_j^2} - \frac{1}{\overline{\rho}}\left[\frac{\partial \overline{P}}{\partial z} + \overline{\rho}\, g\right]$$

If we assume that the mean state is in ***hydrostatic equilibrium*** ($\partial\overline{P}/\partial z = -\overline{\rho}\, g$), then the term in square brackets is zero. Furthermore, if we remember from section 3.3.1 that $\rho'/\overline{\rho}$ is on the order of 3.33×10^{-3} , then we see that the factor on the left hand side of the equation is approximated by $(1 + \rho'/\overline{\rho}) \cong 1$. We can't neglect, however, the first term on the right hand side of the equal sign, because the product $[\rho'/\overline{\rho}\, g]$ is as large as the other terms in the equation. The process of neglecting density variations in the inertia (storage) term, but retaining it in the buoyancy (gravity) term is called the ***Boussinesq approximation***. These two approximations leave

$$\frac{d(\overline{W}+w')}{dt} = -\frac{\rho'}{\overline{\rho}} g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \nu \frac{\partial^2(\overline{W}+w')}{\partial x_j^2}$$

A prerequisite for the Boussinesq approximation is that the shallow convection conditions be satisfied.

By comparing the original equations to the scaled equations above, one finds differences in the terms involving ρ and g. Thus, a simple way to apply the Boussinesq approximation without performing the complete derivation is stated here:

Practical Application of the Boussinesq Approximation:
Given any of the original governing equations, replace every occurrence of ρ with $\overline{\rho}$, and replace every occurrence of g with $\left[\, g - (\theta_v'/\overline{\theta_v})g \,\right]$.

Although subsidence, $\overline{W}$, is important in mass conservation and in the advection of material (moisture, pollutants, etc) from aloft, we see that it is less important in the momentum equation because it is always paired in a linear manner with w'. In fair-weather boundary layers, subsidence can vary from zero to 0.1 m/s, which is considered a relatively large value. This is small compared to the vertical velocity fluctuations, which frequently vary over the range 0 to 5 m/s. Thus, for **only** the momentum equation for fair-weather conditions can we usually ***neglect subsidence:***

$$\overline{W} \equiv 0 \tag{3.3.3a}$$

This leaves the vertical component of the momentum equation as

$$\frac{dw'}{dt} = -\left(\frac{\rho'}{\overline{\rho}}\right) g - \frac{1}{\overline{\rho}} \frac{\partial p'}{\partial z} + \nu \frac{\partial^2 w'}{\partial x_j^2}$$

Using (3.3.1) to replace the density variations with temperature variations gives

$$\frac{dw'}{dt} = \left(\frac{\theta_v'}{\overline{\theta}_v}\right) g - \frac{1}{\overline{\rho}} \frac{\partial p'}{\partial z} + \nu \frac{\partial^2 w'}{\partial x_j^2} \tag{3.3.3b}$$

The physical interpretation of the first two terms in (3.3.3b) is that warmer than average air is accelerated upward (i.e., hot air rises). The last two terms describe the influences of pressure gradients and viscous stress on the motion. This equation therefore plays an important role in the evolution of convective thermals.

Horizontal Component. Although the BL winds are rarely geostrophic, we can use the definition of the ***geostrophic wind*** as a substitute variable for the horizontal pressure gradient terms:

$$f_c U_g = -\frac{1}{\rho} \frac{\partial p}{\partial y} \qquad \text{and} \qquad f_c V_g = +\frac{1}{\rho} \frac{\partial p}{\partial x} \tag{3.3.3c}$$

Thus, the horizontal components of (3.2.3b) become

$$\frac{dU}{dt} = -f_c(V_g - V) + \nu \frac{\partial^2 U}{\partial x_j^2} \tag{3.3.3d}$$

$$\frac{dV}{dt} = +f_c(U_g - U) + \nu \frac{\partial^2 V}{\partial x_j^2} \tag{3.3.3e}$$

I II III

Term I is the ***inertia*** or storage term. Term II is sometimes called the ***geostrophic departure*** term, because it is zero when the actual winds are geostrophic. As we stated before, however, the winds are rarely geostrophic in the BL. Term III describes viscous shear stress.

Combined Momentum Equation. Combining the results from the previous two subsections yields

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\varepsilon_{ij3} f_c (U_{gj} - U_j) + \delta_{i3}\left[\frac{\theta_v'}{\overline{\theta_v}} g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z}\right] + \nu \frac{\partial^2 U_i}{\partial x_j^2} \tag{3.3.3f}$$

where we have applied the shallow convection, incompressibility, hydrostatic and Boussinesq approximations, and where $U_{g_j} = (U_g, V_g, 0)$.

3.3.4 Horizontal Homogeneity

Expanding the total derivative of any mean variable, $\overline{\xi}$, yields

$$\frac{d\overline{\xi}}{dt} = \frac{\partial\overline{\xi}}{\partial t} + U\frac{\partial\overline{\xi}}{\partial x} + V\frac{\partial\overline{\xi}}{\partial y} + W\frac{\partial\overline{\xi}}{\partial z} \tag{3.3.4}$$

I II III IV

From examples like Figs 1.12 and 2.9 we saw that averaged variables such as potential temperature or turbulence kinetic energy exhibit large vertical variations over the 1 to 2 km of boundary layer depth. Those same variables, however, usually exhibit a much smaller horizontal variation over the same 1 to 2 km scale. Counteracting this disparity of gradients is a disparity of velocities. Namely, U and V are often on the order of m/s while

W is on the order of mm/s or cm/s. The resulting terms I through IV in the above equation are thus nearly equal in magnitude for many cases.

The bottom line is that we usually can **not** neglect horizontal advection (terms II & III), and we can **not** neglect subsidence (term IV) as it affects the movement of conserved variables.

Sometimes micrometeorologists wish to focus their attention on turbulence effects at the expense of neglecting mean advection. By assuming ***horizontal homogeneity***, we can set $\partial\overline{\xi}/\partial x = 0$ and $\partial\overline{\xi}/\partial y = 0$, and ***neglecting subsidence*** gives $\overline{W} = 0$. Although these assumptions are frequently made by theorists to simplify their derivations, they are rarely valid in the real atmosphere. When they are made, they cause the advection terms of only mean variables (like $\overline{\xi}$) to disappear; the turbulent flux terms do **not** disappear, and in fact are very important.

3.3.5 Reorientating and Rotating the Coordinate System

Although we usually use a Cartesian coordinate system aligned such that the (x, y, z) axes point (east, north, up), sometimes it is convenient to rotate the Cartesian coordinate system about the vertical (z) axes to cause x and y to point in other directions. Some examples include aligning the x-axis with:

the mean wind direction,
the geostrophic wind direction
the direction of surface stress, or
perpendicular to shorelines or mountains.

The only reason for doing this is to simplify some of the terms in the governing equations. For example, by choosing the x-axis aligned with the mean wind, we find U=M and V=0. In such a system, the x-axis is called the ***along-wind direction*** and the y-direction is called the ***crosswind direction***.

3.4 Equations for Mean Variables in a Turbulent Flow

3.4.1 Equation of State

As was already stated in section 3.3.1, the equation of state is assumed to hold in the mean, and is rewritten here for the sake of organization:

$$\frac{\overline{P}}{\Re} = \overline{\rho}\,\overline{T_v} \qquad (3.4.1)$$

3.4.2 Continuity Equation

Start with the continuity equation (3.2.2c) and expand the velocities into mean and turbulent parts to give:

$$\frac{\partial\,(\overline{U_j} + u_j')}{\partial x_j} = 0$$

or

$$\frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j} = 0 \tag{3.4.2a}$$

Next average over time

$$\overline{\frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j}} = 0$$

Upon applying Reynold's averaging rules, the last term becomes zero, leaving

$$\frac{\partial \overline{U_j}}{\partial x_j} = 0 \tag{3.4.2b}$$

Thus, the continuity equation holds in the mean. Subtracting this from (3.4.2a) gives the continuity equation for turbulent fluctuations:

$$\frac{\partial u_j'}{\partial x_j} = 0 \tag{3.4.2c}$$

This equation will allow us to put turbulent advection terms into flux form, in the same manner as was demonstrated for (3.3.2).

3.4.3 Conservation of Momentum

Starting with the conservation of momentum expressed by (3.2.3b), make the Boussinesq approximation:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3}\left[g - \left(\frac{\theta_v'}{\overline{\theta_v}}\right) g\right] + f_c \varepsilon_{ij3} U_j - \frac{1}{\overline{\rho}} \frac{\partial P}{\partial x_i} + \frac{\nu \partial^2 U_i}{\partial x_j^2}$$

Next, expand the dependent variables into mean and turbulent parts (except for the $\theta_v'/\overline{\theta_v}$ term, for which the expansion has previously been made):

$$\frac{\partial(\overline{U}_i + u_i')}{\partial t} + \frac{(\overline{U}_j + u_j')\partial(\overline{U}_i + u_i')}{\partial x_j} = -\delta_{i3}\left[g - \left(\frac{\theta_v'}{\overline{\theta}_v}\right) g\right] + f_c \varepsilon_{ij3} (\overline{U}_j + u_j')$$

$$- \frac{1}{\overline{\rho}} \frac{\partial(\overline{P}+p')}{\partial x_i} + \nu \partial^2 \frac{(\overline{U}_i + u_i')}{\partial x_j^2}$$

Upon performing the indicated multiplications, and separating terms, we find

$$\frac{\partial \overline{U}_i}{\partial t} + \frac{\partial u_i'}{\partial t} + \frac{\overline{U}_j \partial \overline{U}_i}{\partial x_j} + \frac{\overline{U}_j \partial u'_i}{\partial x_j} + \frac{u_j' \partial \overline{U}_i}{\partial x_j} + \frac{u_j' \partial u_i'}{\partial x_j} =$$

$$-\delta_{i3} g + \delta_{i3}\left(\frac{\theta_v'}{\overline{\theta}_v}\right) g + f_c \varepsilon_{ij3} \overline{U}_j + f_c \varepsilon_{ij3} u_j' - \frac{1}{\overline{\rho}} \frac{\partial \overline{P}}{\partial x_i} - \frac{1}{\overline{\rho}} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 \overline{U}_i}{\partial x_j^2} + \nu \frac{\partial^2 u_i'}{\partial x_j^2}$$

(3.4.3a)

Next, average the whole equation:

$$\overline{\frac{\partial \overline{U}_i}{\partial t}} + \overline{\frac{\partial u_i'}{\partial t}} + \overline{\frac{\overline{U}_j \partial \overline{U}_i}{\partial x_j}} + \overline{\frac{\overline{U}_j \partial u'_i}{\partial x_j}} + \overline{\frac{u_j' \partial \overline{U}_i}{\partial x_j}} + \overline{\frac{u_j' \partial u_i'}{\partial x_j}} =$$

$$\overline{- \delta_{i3} g} + \overline{\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta}_v}\right) g} + \overline{f_c \varepsilon_{ij3} \overline{U}_j} + \overline{f_c \varepsilon_{ij3} u_j'} - \overline{\frac{1}{\overline{\rho}} \frac{\partial \overline{P}}{\partial x_i}} - \overline{\frac{1}{\overline{\rho}} \frac{\partial p'}{\partial x_i}} + \overline{\nu \frac{\partial^2 \overline{U}_i}{\partial x_j^2}} + \overline{\nu \frac{\partial^2 u_i'}{\partial x_j^2}}$$

By applying Reynolds averaging rules the second, fourth, fifth, eighth, tenth, twelfth and fourteenth terms become zero. We are left with:

$$\frac{\partial \overline{U}_i}{\partial t} + \frac{\overline{U}_j \partial \overline{U}_i}{\partial x_j} + \frac{\overline{u_j' \partial u_i'}}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} \overline{U}_j - \frac{1}{\overline{\rho}} \frac{\partial \overline{P}}{\partial x_i} - \frac{\nu \partial^2 \overline{U}_i}{\partial x_j^2} \quad (3.4.3b)$$

Finally, multiply the continuity equation for turbulent motions (3.4.2c) by u_i', average it, and add it to (3.4.3b) to put the turbulent advection term into flux form:

$$\frac{\partial \overline{U_i}}{\partial t} + \frac{\overline{U_j}\partial \overline{U_i}}{\partial x_j} + \frac{\partial(\overline{u_i'u_j'})}{\partial x_j} = -\delta_{i3}g + f_c\varepsilon_{ij3}\overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \frac{\nu\partial^2\overline{U_i}}{\partial x_j^2}$$

By moving this flux term to the right hand side of the equation, we see something very remarkable; namely, the following forecast equation for mean wind is very similar to the basic conservation equation we started with (3.2.3b), except for the addition of the turbulence term at the end.

$$\underset{\text{I}}{\frac{\partial \overline{U_i}}{\partial t}} + \underset{\text{II}}{\overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j}} = \underset{\text{III}}{-\delta_{i3}g} + \underset{\text{IV}}{f_c\varepsilon_{ij3}\overline{U_j}} \underset{\text{V}}{- \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i}} + \underset{\text{VI}}{\frac{\nu\partial^2\overline{U_i}}{\partial x_j^2}} \underset{\text{X}}{- \frac{\partial(\overline{u_i'u_j'})}{\partial x_j}} \tag{3.4.3c}$$

Term I represents storage of mean momentum (inertia).
Term II describes advection of mean momentum by the mean wind.
Term III allows gravity to act in the vertical direction only.
Term IV describes the influence of the earth's rotation (Coriolis effects).
Term V describes the mean pressure-gradient forces.
Term VI represents the influence of viscous stress on the mean motions.
Term X represents the influence of Reynolds' stress on the mean motions (see section 2.9.2). It can also be described as the divergence of turbulent momentum flux.

Term X can also be written as $(1/\overline{\rho})\ \partial\tau_{ij\,\text{Reynolds}}/\partial x_j$ where $\tau_{ij\,\text{Reynolds}} = -\overline{\rho}\ \overline{u_i'\,u_j'}$. The implication of this last term is that **turbulence must be considered** in making forecasts in the turbulent boundary layer, **even if we are trying to forecast only mean quantities**. Term X can often be as large in magnitude, or larger, than many other terms in the equation. Sometimes term X is labled as "F" by large-scale dynamists to denote friction.

3.4.4 Conservation of Moisture

For total specific humidity, start with (3.2.4a) and split the dependent variables into mean and turbulent parts:

$$\frac{\partial \overline{q_T}}{\partial t} + \frac{\partial q_T'}{\partial t} + \frac{\overline{U_j}\,\partial \overline{q_T}}{\partial x_j} + \frac{\overline{U_j}\,\partial q_T'}{\partial x_j} + \frac{u_j'\,\partial \overline{q_T}}{\partial x_j} + \frac{u_j'\,\partial q_T'}{\partial x_j} =$$

$$\frac{\nu_q \partial^2 \overline{q}}{\partial x_j^2} + \frac{\nu_q \partial^2 q'}{\partial x_j^2} + \frac{S_{qT}}{\overline{\rho}_{air}} \tag{3.4.4a}$$

where the net remaining source term, S_{qT}, is assumed to be a mean forcing. Next, average the equation, apply Reynolds' averaging rules, and use the turbulent continuity equation to put the turbulent advection term into flux form:

$$\frac{\partial \overline{q_T}}{\partial t} + \frac{\overline{U_j}\,\partial \overline{q_T}}{\partial x_j} = \frac{\nu_q \partial^2 \overline{q}}{\partial x_j^2} + \frac{S_{qT}}{\overline{\rho}_{air}} - \frac{\partial(\overline{u_j' q_T'})}{\partial x_j} \tag{3.4.4b}$$

$$\text{I} \qquad \text{II} \qquad \text{VI} \qquad \text{VII} \qquad \text{X}$$

Term I represents the storage of mean total moisture.
Term II describes the advection of mean total moisture by the mean wind.
Term VI represents the mean molecular diffusion of water vapor.
Term VII is the mean net body source term for additional moisture processes.
Term X represents the divergence of turbulent total moisture flux.

As before, this equation is similar to the basic conservation equation (3.2.4a), except for the addition of the turbulence term at the end. Similar equations can be written for the vapor and non-vapor parts of total specific humidity.

3.4.5 Conservation of Heat

Start with the basic heat conservation equation (3.2.5) and expand the dependent variables into mean and turbulent parts

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\partial \theta'}{\partial t} + \frac{\overline{U_j}\,\partial \overline{\theta}}{\partial x_j} + \frac{\overline{U_j}\,\partial \theta'}{\partial x_j} + \frac{u_j'\,\partial \overline{\theta}}{\partial x_j} + \frac{u_j'\,\partial \theta'}{\partial x_j} =$$

$$\frac{\nu_\theta \partial^2 \overline{\theta}}{\partial x_j^2} + \frac{\nu_\theta \partial^2 \theta'}{\partial x_j^2} - \frac{1}{\overline{\rho}\, C_p} \frac{\partial \overline{Q_j^*}}{\partial x_j} - \frac{1}{\overline{\rho}\, C_p} \frac{\partial Q_j^{*\prime}}{\partial x_j} - \frac{L_v E}{\overline{\rho}\, C_p} \tag{3.4.5a}$$

Next, Reynolds average and put the turbulent advection term into flux form to give:

$$\frac{\partial\overline{\theta}}{\partial t} + \frac{\overline{U}_j\,\partial\overline{\theta}}{\partial x_j} = \frac{\nu_\theta\,\partial^2\overline{\theta}}{\partial x_j^2} - \frac{1}{\overline{\rho}\,C_p}\frac{\partial\overline{Q_j^*}}{\partial x_j} - \frac{L_v\,E}{\overline{\rho}\,C_p} - \frac{\partial(\overline{u_j'\theta'})}{\partial x_j} \qquad (3.4.5b)$$

I II VI VII VIII X

Term I represents the mean storage of heat.
Term II describes the advection of heat by the mean wind.
Term VI represents the mean molecular conduction of heat.
Term VII is the mean net body source associated with radiation divergence.
Term VIII is the body source term associated with latent heat release.
Term X represents the divergence of turbulent heat flux.

3.4.6 Conservation of a Scalar Quantity

Start with the basic conservation equation (3.2.6) of tracer C and expand into mean and turbulent parts:

$$\frac{\partial\overline{C}}{\partial t} + \frac{\partial c'}{\partial t} + \frac{\overline{U}_j\partial\overline{C}}{\partial x_j} + \frac{\overline{U}_j\partial c'}{\partial x_j} + \frac{u_j'\,\partial\overline{C}}{\partial x_j} + \frac{u_j'\partial c'}{\partial x_j} =$$

$$\frac{\nu_c\partial^2\overline{C}}{\partial x_j^2} + \frac{\nu_c\partial^2 c'}{\partial x_j^2} + S_c \qquad (3.4.6a)$$

where the net remaining source term, S_c, is assumed to be a mean forcing. Next, Reynolds average and use the turbulent continuity equation to put the turbulent advection term into flux form:

$$\frac{\partial\overline{C}}{\partial t} + \frac{\overline{U}_j\,\partial\overline{C}}{\partial x_j} = \frac{\nu_c\partial^2\overline{C}}{\partial x_j^2} + S_c - \frac{\partial(\overline{u_j'c'})}{\partial x_j} \qquad (3.4.6b)$$

I II VI VII X

Term I represents the mean storage of tracer C.
Term II describes the advection of the tracer by the mean wind.
Term VI represents the mean molecular diffusion of the tracer.
Term VII is the mean net body source term for additional tracer processes.
Term X represents the divergence of turbulent tracer flux.

3.5 Summary of Equations, with Simplifications

To simplify usage of the equations, we have collected them in this section and organized them in a way that similarities and differences can be more easily noted. Before we list these equations, however, we can make one additional simplification based on the scale of viscous effects vs. turbulent effects on the mean fields.

3.5.1 The Reynolds Number

The ***Reynolds number***, Re, is defined as

$$\mathrm{Re} \equiv \mathbb{V}\,\mathbb{L}/\nu = \rho\,\mathbb{V}\,\mathbb{L}/\mu \tag{3.5.1}$$

where $\mathbb{V}$ and $\mathbb{L}$ are velocity and length scales in the boundary layer. Given $\nu_{air} \cong 1.5 \times 10^{-5}\ m^2 s^{-1}$ and the typical scaling values $\mathbb{V} = 5$ m/s and $\mathbb{L} = 100$ m in the surface layer, we find that $\mathrm{Re} = 3 \times 10^7$. In the atmospheric mixed layer, the Reynolds number is even larger. The Reynolds number can be interpreted as the ratio of inertial to viscous forcings.

3.5.2 Neglect of Viscosity for Mean Motions

In each of the conservation equations except mass conservation, there are molecular diffusion/viscosity terms. Observations in the atmosphere indicate that the molecular diffusion terms are several order of magnitudes smaller than the other terms and can be neglected.

For example, after making the hydrostatic assumption, the momentum conservation equation for mean motions in turbulent flow (3.4.3c) can be rewritten as

$$\left[\frac{\partial \overline{U_i}}{\partial t}\right] + \left[\overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j}\right] = \left[f_c\,\varepsilon_{ij3}\,\overline{U_j}\right] - \left[\frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x}\right] - \left[\frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial y}\right] - \left[\frac{\partial \overline{u_i' u_j'}}{\partial x_j}\right] + \frac{1}{\mathrm{Re}}\left[\mathbb{V}\mathbb{L}\,\frac{\partial^2 \overline{U_i}}{\partial x_j^2}\right] \tag{3.5.2}$$

Each of the terms in square brackets is roughly the same order of magnitude. The last term, however, is multiplied by (1/Re) -- a very small number (on the order of 10^{-7}). Hence, the last term can be neglected compared to the rest, except in the lowest few centimeters above the surface.

3.5.3 Summary of Equations for Mean Variables in Turbulent Flow

Neglecting molecular diffusion and viscosity, and making the hydrostatic and Boussinesq approximations to the governing equations leaves:

$$\frac{\overline{P}}{\Re} = \overline{\rho}\,\overline{T_v} \tag{3.5.3a}$$

$$\frac{\partial \overline{U_j}}{\partial x_j} = 0 \tag{3.5.3b}$$

$$\frac{\partial \overline{U}}{\partial t} + \overline{U_j}\frac{\partial \overline{U}}{\partial x_j} = -f_c(\overline{V_g} - \overline{V}) - \frac{\partial(\overline{u_j'u'})}{\partial x_j} \tag{3.5.3c}$$

$$\frac{\partial \overline{V}}{\partial t} + \overline{U_j}\frac{\partial \overline{V}}{\partial x_j} = +f_c(\overline{U_g} - \overline{U}) - \frac{\partial(\overline{u_j'v'})}{\partial x_j} \tag{3.5.3d}$$

$$\frac{\partial \overline{q_T}}{\partial t} + \overline{U_j}\frac{\partial \overline{q_T}}{\partial x_j} = +S_{qT}/\overline{\rho_{air}} - \frac{\partial(\overline{u_j'q_T'})}{\partial x_j} \tag{3.5.3e}$$

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{U_j}\frac{\partial \overline{\theta}}{\partial x_j} = -\frac{1}{\overline{\rho}\,C_p}\left[L_v E + \frac{\partial \overline{Q_j^*}}{\partial x_j}\right] - \frac{\partial(\overline{u_j'\theta'})}{\partial x_j} \tag{3.5.3f}$$

$$\frac{\partial \overline{C}}{\partial t} + \overline{U_j}\frac{\partial \overline{C}}{\partial x_j} = +S_c - \frac{\partial(\overline{u_j'c'})}{\partial x_j} \tag{3.5.3g}$$

I II VII X

The similarity between the last five equations reflects that the same forcings are present in each conservation equation:

Term I represents storage.
Term II represents advection.
Term VII represent sundry body forcings.
Term X describes the turbulent flux divergence.

The covariances appearing in term X reinforce the earlier assertion that statistics play an important role in the study of turbulent flow.

In the two momentum equations above, the ***mean geostrophic wind components*** were defined using the mean horizontal pressure gradients:

$$\overline{U_g} = -\frac{1}{f_c \overline{\rho}} \frac{\partial \overline{P}}{\partial y} \qquad \text{and} \qquad \overline{V_g} = +\frac{1}{f_c \overline{\rho}} \frac{\partial \overline{P}}{\partial x} \tag{3.5.3h}$$

Sometimes the left hand side of equations c thru g are simplified using

$$\frac{d()}{dt} = \frac{\partial()}{\partial t} + \frac{\overline{U_j}\,\partial()}{\partial x_j} \tag{3.5.3i}$$

where the total derivative d( )/dt is inferred to include only ***mean*** advective effects, and not the turbulent effects.

3.5.4 Examples

Many applications will have to wait until more realistic PBL initial and boundary conditions have been covered. For now, just a few artificial sample exercises showing the use of equations (3.5.3) will be presented.

Problem 1. Suppose that the turbulent heat flux decreases linearly with height according to $\overline{w'\theta'} = a - b\,z$, where a = 0.3 (K ms^{-1}) and b = $3x10^{-4}$ (K s^{-1}). If the initial potential temperature profile is an arbitrary shape (i.e., pick a shape), then what will be the shape of final profile one hour later? Neglect subsidence, radiation, latent heating, and assume horizontal homogeneity.

Solution. Neglecting subsidence, radiation, and latent heating leaves (3.5.3f) as

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\overline{U}\partial \overline{\theta}}{\partial x} + \frac{\overline{V}\partial \overline{\theta}}{\partial y} = -\frac{\partial(\overline{u'\theta'})}{\partial x} - \frac{\partial(\overline{v'\theta'})}{\partial y} - \frac{\partial(\overline{w'\theta'})}{\partial z}$$

By assuming horizontal homogeneity, the x and y derivatives drop out, giving

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial(\overline{w'\theta'})}{\partial z}$$

Plugging in the expression for $\overline{w'\theta'}$ gives $\partial\overline{\theta}/\partial t = +b$. This answer is not a function of z; hence, air at each height in the sounding warms at the same rate. Integrating over time from t = t_o to t gives

$$\overline{\theta}\,\big|_T = \overline{\theta}\big|_{to} + b(t - t_0)$$

The warming in one hour is b (t - t_o) = [$3x10^{-4}$ (K/s)]·[3600 (s)] = 1.08 K.

Discussion. This scenario frequently occurs in daytime mixed layers. Thus, given an adiabatic ML initially, the potential temperature profile a bit later will also be adiabatic because air at all heights is warming at the same rate. In fact, ***anytime the heat flux changes linearly with height, the shape of the potential temperature profile will be preserved while it warms, regardless of its initial shape.***

__Problem 2__. If a horizontal wind of 10 m/s is advecting drier air into a region, where the horizontal moisture gradient is (5 g_{water}/kg_{air})/100 km, then what vertical gradient of turbulent moisture flux in the BL is required to maintain a steady-state specific humidity? Assume all the water is in vapor form, and that there is no body source of moisture. Be sure to state any additional assumptions you make.

Solution. A ***steady-state*** situation is defined as one where there are no local changes of a variable with time (i.e., where ∂( )/∂t = 0). Choose the x-axis to be aligned with the mean wind direction for simplicity. Equation (3.5.3e) becomes

$$\frac{\overline{U}\partial\overline{q}}{\partial x} + \frac{\overline{W}\partial\overline{q}}{\partial z} = -\frac{\partial(\overline{u'q'})}{\partial x} - \frac{\partial(\overline{v'q'})}{\partial y} - \frac{\partial(\overline{w'q'})}{\partial z}$$

No information was given in the problem about subsidence, or about horizontal flux gradients; therefore, for simplicity let's assume they are zero. This leaves

$$\frac{\overline{U}\partial\overline{q}}{\partial x} = -\frac{\partial(\overline{w'q'})}{\partial z}$$

$$[10\ (m/s)]\cdot[\ 5\times10^{-5}\ (\ g_{water}\ \ (kg_{air}\ )^{-1}m^{-1}\ )] = -\frac{\partial(\overline{w'q'})}{\partial z}$$

Thus

$$\frac{\partial(\overline{w'q'})}{\partial z} = -5\times10^{-4}\ g_{water}\ (kg_{air}\ )^{-1}s^{-1}$$

Discussion. A gradient of this magnitude corresponds to a 0.5 (g/kg)(m/s) decrease of $\overline{w'q'}$ over a vertical distance of 1 km. Also, we see from both sample problems that a decrease of turbulence flux with height (i.e., flux convergence) results in an increase of the mean variable (e.g., temperature or moisture) with time. For the latter example, the potential increase was balanced by advective drying.

__Problem 3__: Assume a turbulent BL at a latitude of 44°N, where the mean wind is 2 m/s slower than geostrophic (i.e., the wind is subgeostrophic). Neglect subsidence, and assume horizontal homogeneity and steady state.

a) Find the Reynolds stress divergence necessary to support this velocity deficit.

b) If that stress divergence was related to molecular viscosity instead of turbulence, what curvature in the mean wind profile would be necessary?

Solution. a) Pick a coordinate system aligned with the stress, for simplicity. Therefore, use equation (3.5.3c). Assuming horizontal homogeneity, steady state, and neglecting subsidence leaves

$$0 = -f_c(\overline{V}_g - \overline{V}) - \frac{\partial \overline{u'w'}}{\partial z}$$

or

$$-\frac{\partial \overline{u'w'}}{\partial z} = f_c(\overline{V}_g - \overline{V}) = [10^{-4}(s^{-1})]\,[2\ (m/s)] = 2 \times 10^{-4} m\ s^{-2}$$

b) Looking back at equation (3.4.3c), we see that the viscous stress term is expressed by $\nu\partial^2U/\partial z^2$. Thus $\nu\partial^2U/\partial z^2 = 2\text{x}10^{-4}$ m s^{-2}. Using the value of ν from appendix C, we can solve for the wind profile curvature $\partial^2U/\partial z^2$:

$$\frac{\partial^2\overline{U}}{\partial z^2} = \frac{[2 \times 10^{-4}(m\ s^{-2})]}{[1.5 \times 10^{-5}(m^2 s^{-1})]} = 13.33\ (m\ s)^{-1}$$

Discussion. This is a tremendously large value for curvature. If we assume that such a profile was observed within the the middle of the BL (z=0.5z_i), where the wind speed is, say 5 m/s, then we can integrate the above equation to find the mean wind at any other height z' away from the middle of the BL: $U(z = 0.5z_i+z') = 5 + 6.67\ z'^2$. For example, at a height of 0.5z_i + 10m, the wind speed would be 672 m/s, assuming no shear at z_i. Since realistic wind speeds and shears are several orders of magnitude smaller over most of the PBL, it is apparent that viscous stress plays a much smaller role than turbulent Reynolds' stress in the mean wind equation. As we shall see later, however, ***viscosity is very important for turbulent motions, and can not be neglected.***

3.6 Case Studies

3.6.1 Daytime Cases

The following cases are meant to acquaint the reader with typical observations of some of the terms in the equations of this chapter. They are analyses of real data, most of which are based on the BLX83 field experiment near Chickasha, Oklahoma (Stull and Eloranta, 1984). This data set was taken in fair-weather anticyclonic conditions during the daytime when deep convective mixed layers formed.

Figs 3.1a through 3.3a show heat and moisture fluxes as measured by an instrumented Queen Air aircraft, flying at about 72 m/s along level flight paths of about 30 km long. Measurements of w, T, and q were taken 20 times per second [i.e., two measurements per 7.2 m (this is the ***Nyquist wavelength***, discussed in chapter 8)]. From this data, average values over each flight leg were found, and linear trends were calculated. These were subtracted from the observed values to give w', T', and q'.

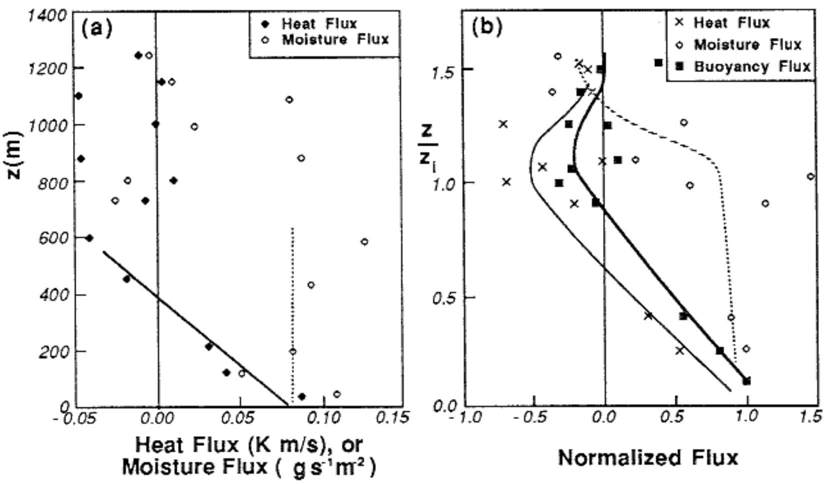

Fig. 3.1 (a) Kinematic heat flux ($\overline{w'\theta'}$) and moisture flux ($\overline{w'\rho_v'}$) at various heights measured by aircraft during flight 2, May 27, of the 1983 Boundary Layer Experiment (BLX83) in Oklahoma, where ρ_v is the water vapor density ($g_{water}\, m^{-3}_{air}$). (b) Same data as (a), but normalized with respect to the top of the mixed layer (z_i), and with respect to surface values of the fluxes. The buoyancy flux ($\overline{w'\theta_v'}$) is also plotted.

An ***FFT*** (Fast Fourier Transform) filter was used to eliminate all wavelengths longer than 6.25 km. from these space series. This was necessary to reduce the effect of unresolved long (mesoscale) waves that would otherwise contaminate the data. The resulting filtered values were used to calculate kinematic fluxes $\overline{w'T'}$ and $\overline{w'q'}$ using the ***eddy correlation method*** (i.e., the method of exercise (2) in section 2.12; also see chapter 10). Thus, the averages are line (spatial) averages, not time averages. Although each level flight leg took less than 5 min to fly, the many legs making up any one flight took from 2 to 4 hours to complete. The following table list the flight times, where CDT denotes Central Daylight Time (CDT = UTC - 5 h):

Table 3-1. Flight information for selected flights during the BLX83 field experiment.

Flight	Date	Start (CDT)	Duration (hr)	Boundary Layer
2	27 May 1983	1034	2.5	ML
3	28 May 1983	1425	3.6	ML
13	14 June 1983	1406	3.3	ML

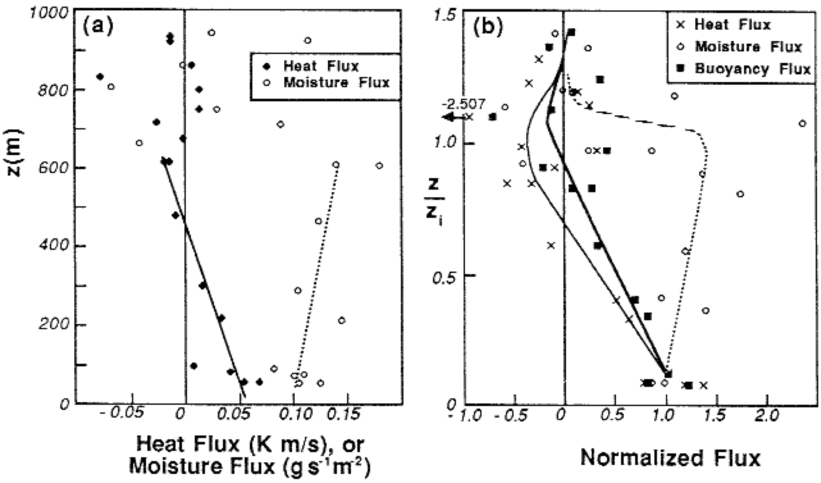

Fig. 3.2 (a) Kinematic heat flux ($\overline{w'\theta'}$) and moisture flux ($\overline{w'\rho_v'}$) at various heights measured by aircraft during flight 3, May 28, of the 1983 Boundary Layer Experiment (BLX83) in Oklahoma, where ρ_v is the water vapor density ($g_{water}\ m^{-3}_{air}$). (b) Same data as (a), but normalized with respect to the top of the mixed layer (z_i), and with respect to surface values of the fluxes. The buoyancy flux ($\overline{w'\theta_v'}$) is also plotted.

Each data point in Figs 3.1a to 3.3a represent a flight leg average flux. In general, we see that the heat flux decreases with height, starting at a large positive value near the surface, and becoming negative near the top of the mixed layer. The positive heat flux near the surface is associated with solar heating of the earth's surface, which transfers its heat to the atmosphere. The negative heat flux near the ML top is associated with the entrainment of warmer FA air down into the ML (warm air mixed down causes a negative heat flux). This slope of the heat flux profile causes the temperature to become warmer with time (see eq 3.5.3f).

There is much more scatter in the moisture flux values in these figures. In general, they are positive near the surface, implying evaporation of moisture from the ground into the air. The values just below the top of the ML are also positive, which in this case is related to dry air being entrained down into the ML (note that moist air moving up and dry air moving down both yield a positive moisture flux — see section 2.7). Thus, the moisturizing from the surface and drying from aloft nearly counteract each other in the cases studied, as indicated by the nearly vertical profile of $\overline{w'q'}$ with height (i.e., zero slope implies zero humidity increase, according to eq 3.5.3e). Notice that on Flight 3, the moisture flux increases slightly with height, implying a net drying of the ML.

One problem with these figures is that sufficient time elapses between the low altitude flights and the high altitude flights that non-stationarity of the ML comes into play. In particular, the diurnal cycle causes changes in solar heating with time. Also, the top of the

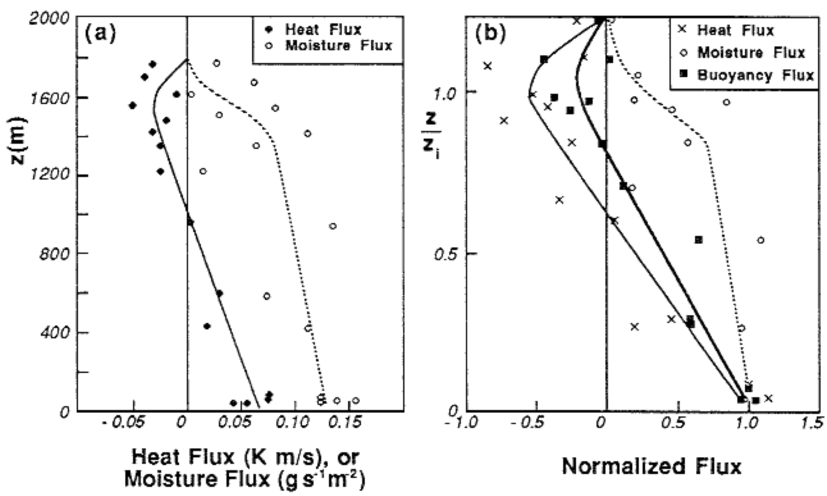

Fig. 3.3 (a) Kinematic heat flux ($\overline{w'\theta'}$) and moisture flux ($\overline{w'\rho_v'}$) at various heights measured by aircraft during flight 13, June 14, of the 1983 Boundary Layer Experiment (BLX83) in Oklahoma, where ρ_v is the water vapor density ($g_{water}\ m^{-3}_{air}$). (b) Same data as (a), but normalized with respect to the top of the mixed layer (z_i), and with respect to surface values of the fluxes. The buoyancy flux ($\overline{w'\theta_v'}$) is also plotted.

ML can increase substantially during that time interval. To remove these effects, micrometeorologist often normalize their plots. Height is normalized by dividing by the depth of the ML, z_i. Flux is normalized by the concurrent surface value of flux observed (or estimated).

The resulting normalized flux profiles are shown in Figs 3.1b - 3.3b. In addition, the ***buoyancy flux*** $\overline{w'\theta_v'}$ is shown. The buoyancy flux has much less scatter than the other fluxes. It is largest at the surface, and decreases linearly with height in the ML.

Figs 3.4a-c show the estimated evolution of the potential temperature profile for the three cases (Crum, et al., 1987). Each data point corresponds to a flight-leg average. The two profiles shown for flight 3 were started near the surface, and each ended an hour later above the top of the ML. Thus, the warming that took place during each hour contaminates the profiles, causing them to appear tilted. The top of each profile is tilted towards the warmer tempeartures. Just the opposite tilt is observed for the flight 13, because the flights above the top of the ML were flown first. In spite of these tilts, it is obvious that the ML becomes warmer and deeper with time.

Better examples of ML evolution for flight 3 are shown in Figs 3.5 and 3.6. The data set in these figures was taken while the aircraft climbed or descended, thereby making soundings. Given typical descent rates of the aircraft, measurements were made with about 0.5 m resolution in the vertical (extremely high resolution soundings).

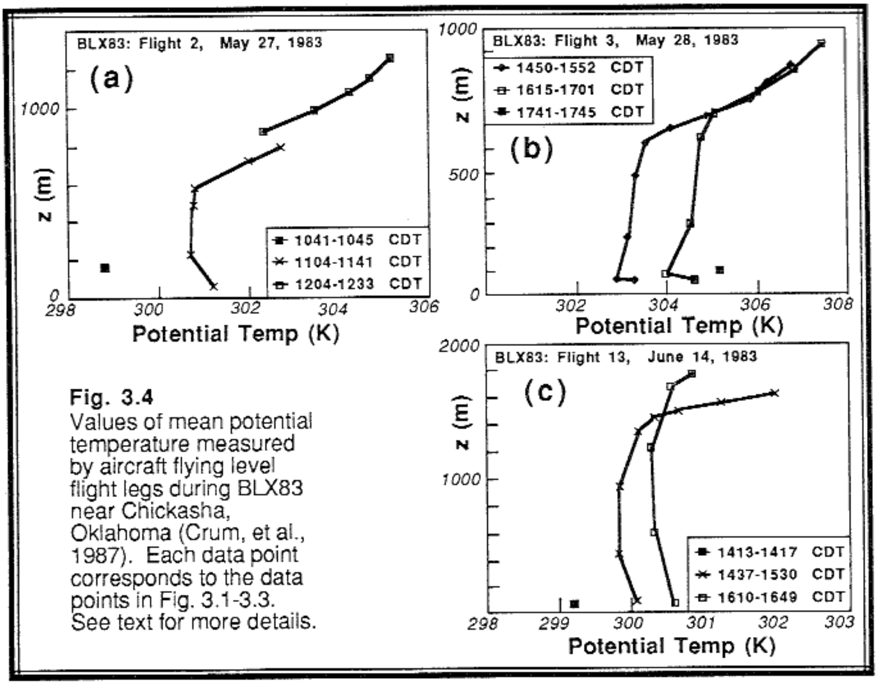

Fig. 3.4
Values of mean potential temperature measured by aircraft flying level flight legs during BLX83 near Chickasha, Oklahoma (Crum, et al., 1987). Each data point corresponds to the data points in Fig. 3.1-3.3. See text for more details.

It took about 10 minutes to complete a sounding, so we can essentially consider the soundings to be instantaneous. These sounding legs were started at the following times:

Leg 1 - 1438 CDT
Leg 12 - 1604
Leg 22 - 1731

The top of the ML, z_i, is very evident by the strong temperature inversion and drop in humidity. ML growth stands out, as does the warming of the ML. There is little change in the humidity with time, however.

From Figs 3.5a-c, we can observe the rate of warming, at any height. This can be compared with the slope of the heat flux profile from Fig 3.2. It is left as an exercise to use (3.5.3f) to see what percentage of the warming within the ML is associated with turbulent flux divergence (convergence) and what percentage is associated with other forcings (radiative, latent heating, advective, etc.).

A similar study can be made for moisture, using the specific humidity evolution shown in Figs 3.6a-c, and comparing that to the expected moisturizing using the moisture flux profiles of Fig 3.2 along with (3.5.3e). For both moisture and temperature, it is evident that the turbulence term in equations (3.5.3) plays a very important role during daytime conditions over land, when vigorous convective mixing is occurring.

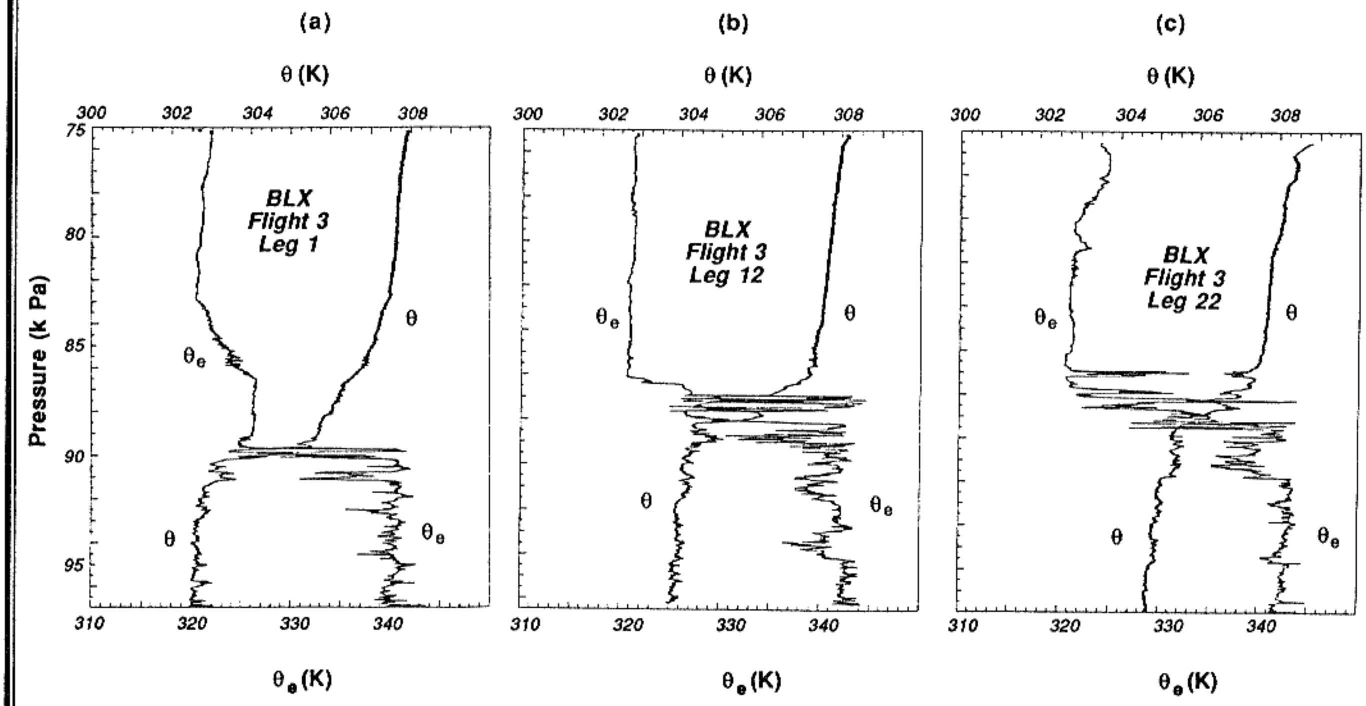

Fig. 3.5 Potential temperature (θ) and equivalent potential temperature (θ_e) soundings made on 28 May 1983 during BLX83 (Crum, et al.,1987) by flying an instrumented aircraft in ascending or descending paths. (a) Leg 1: 1438 CDT. (b) Leg 12: 1604 CDT. (c) Leg 22: 1731 CDT.

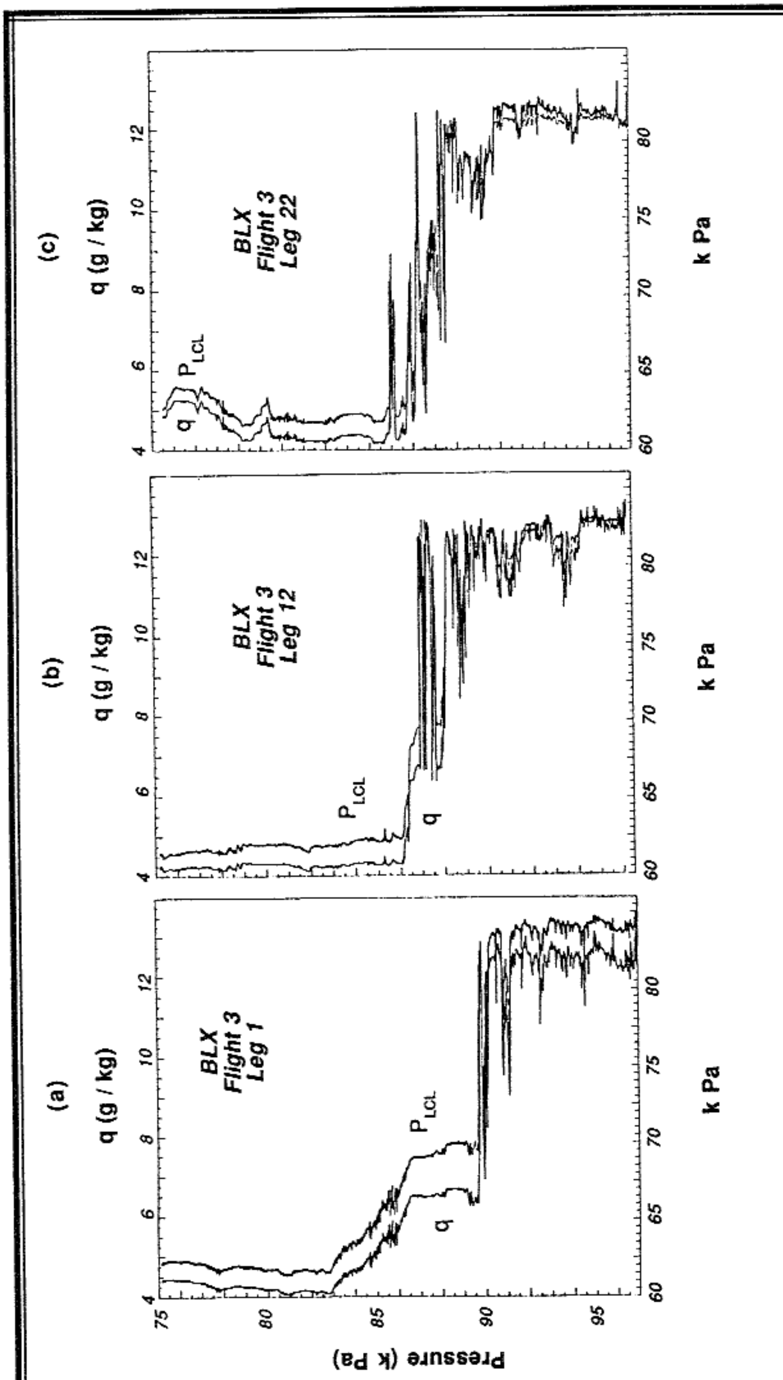

Fig. 3.6 Soundings of specific humidity (q) and pressure of the lifting condensation level (P_{LCL}) made on 28 May 1983 during BLX83 (Crum, et al., 1987) by flying an instrumented aircraft in ascending or descending paths. (a) Leg 1: 1438 CDT. (b) Leg 12: 1604 CDT. (c) Leg 22: 1731 CDT.

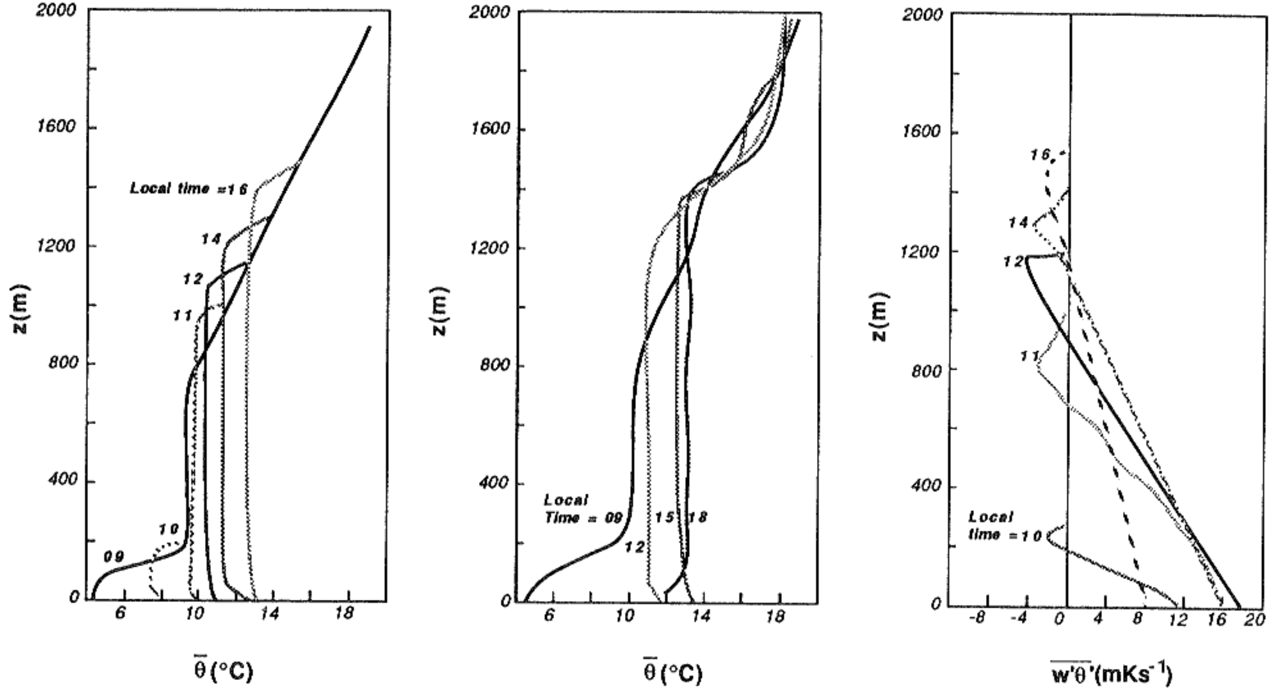

Fig. 3.7 Daytime temperature evolution as a function of local time. Modeled (left) and observed (center) profiles of mean virtual potential temperature during Day 33 of the Wangara field experiment. Computed (right) profiles of kinematic heat flux during Day 33. The computed profiles (André, et al., 1978) show the idealized shape and evolution.

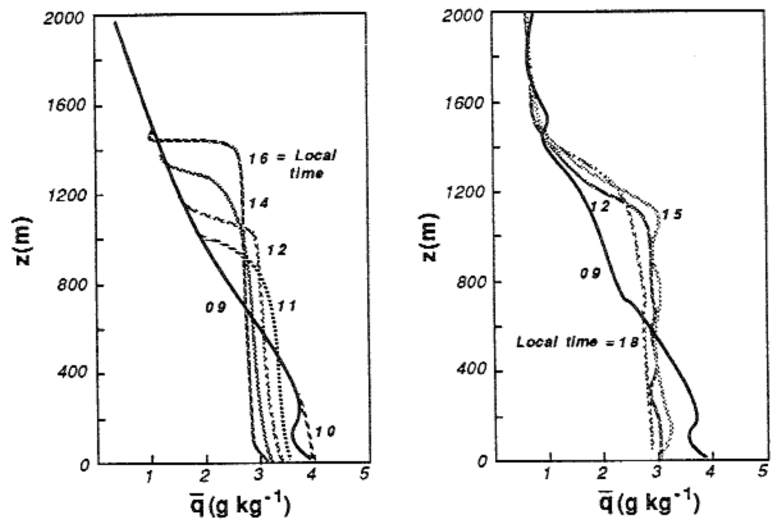

Fig. 3.8 Modeled (left) and observed (right) profiles of mean humidity during Day 33 for Wangara, valid at the local time indicated (André, et al., 1978).

The 1967 Wangara field experiment in Australia (Clarke, et al., 1971) also yielded much useful boundary layer data. André, et al., (1978) have used Day 33 from that experiment as the basis for a numerical simulation of boundary layer evolution. Fig 3.7 shows the modeled $\overline{\theta}$ evolution and the corresponding verification soundings. Modeled heat fluxes are shown in Fig 3.7c. The nearly uniform potential temperature with height is apparent, as are the linear heat flux profiles that are a characteristic signature of convective ML turbulence.

Modeled and observed humidity profiles are shown in Fig 3.8 for the Wangara experiment. The mean specific humidity decreases slightly with height. This slight slope occurs when dry air is entraining into the top of the ML, while moisture is evaporating into the bottom.

Evolution of the observed wind speeds are shown in Fig 3.9. During the afternoon hours when the mixed layer is over 1000 m thick, the winds within the interior of the ML have approximately constant wind speed with height. In the surface layer the winds must decrease towards zero at the ground. Across the entrainment zone at the top of the ML the winds change to their geostrophic values. For these cases, baroclinicity caused the geostrophic wind speed in the mixed layer to be faster than those higher above the ground. As a result, the wind speed above the ML is less than the winds within the ML, even though the ML winds are subgeostrophic and the FA winds are close to geostrophic.

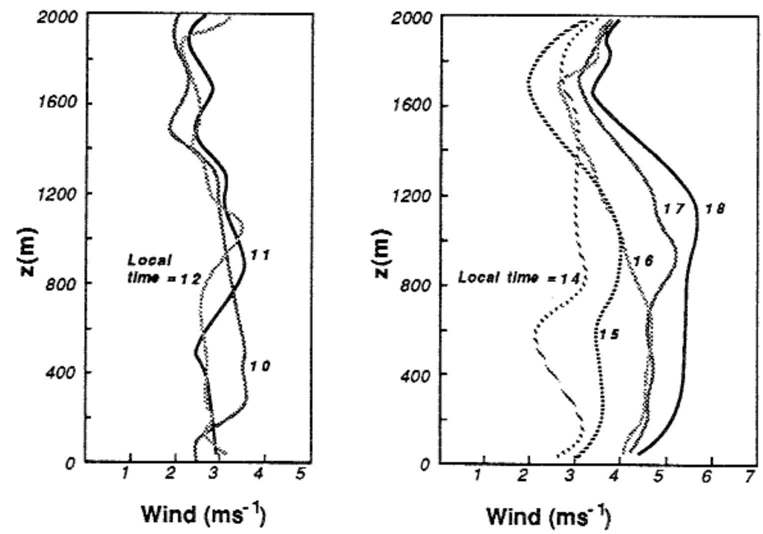

Fig. 3.9 Observed profiles of mean horizontal wind during Day 33, Wangara (Clarke, et al., 1971).

3.6.2 Nighttime

At nighttime the turbulence is often less vigorous. As a result, other effects such as advection, radiation, and subsidence become as important or more important than turbulence in causing changes in temperature and humidity. For example, Fig 3.10 shows BLX83 field experiment data taken during the night of 18 June 1983 near Canton, Oklahoma (Carlson and Stull, 1986). Part (a) shows the temperature evolution between 2100 CDT (plotted as circles) and 2230 CDT (plotted as squares), as observed by special high resolution rawinsonde balloon soundings. Cooling is evident near the surface during this 1.5 hour period, while there is warming aloft.

Also during the night, measurements were made of radiation budgets and subsidence. Computer models were then used to estimate the contributions of the terms in (3.5.3f) towards the total cooling/heating. These contributions are shown in part (b), where the grey lines represent the turbulence part, the dotted line represents radiation divergence, and the solid lines represent subsidence contributions. For the grey and the solid lines, two curves are shown to indicate how they evolved with time between the initial and the final soundings. It is apparent that subsidence and radiation dominate in the upper part of these sounding, but turbulence becomes more important near the ground.

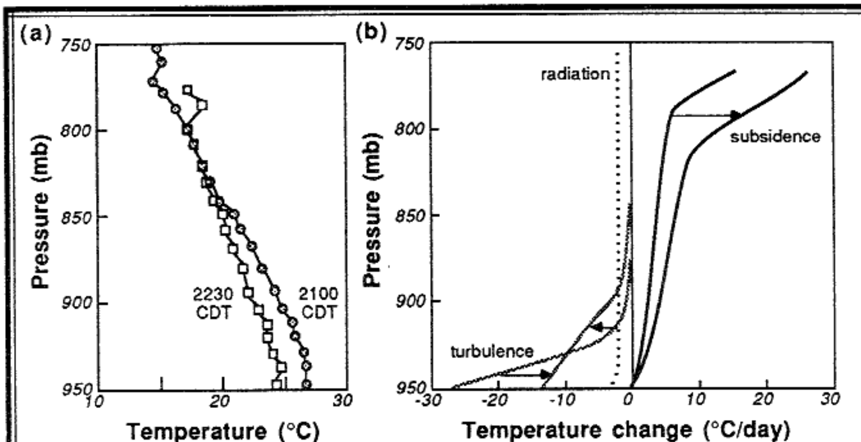

Fig. 3.10 (a) Temperature profiles measured by high resolution rawinsonde soundings launched near Canton, Oklahoma on 18 June 1983 during the BLX 83 field experiment. The circles show data from the 2100 CDT lauch. Squares show those of the 2230 CDT launch. (b) Contributions of turbulence (shaded lines), radiation (dotted line), and subsidence (solid lines) to the cooling rate are modeled for that 1.5 hour period. The two shaded and two solid lines show the range of the respective contributions during the period (Carlson and Stull, 1986).

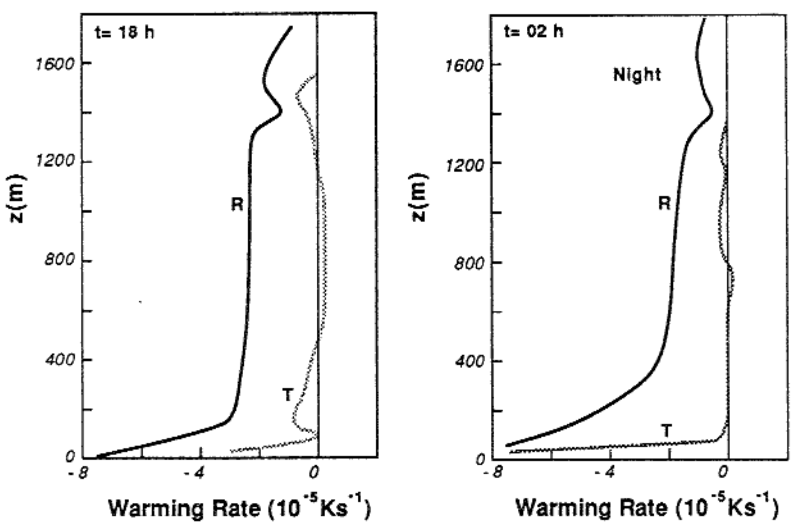

Fig. 3.11 Temperature budgets at t=18 h and t=02 h during Night 33-34 of Wangara: R= radiative transfer; T= turbulent transport (André, et al., 1978).

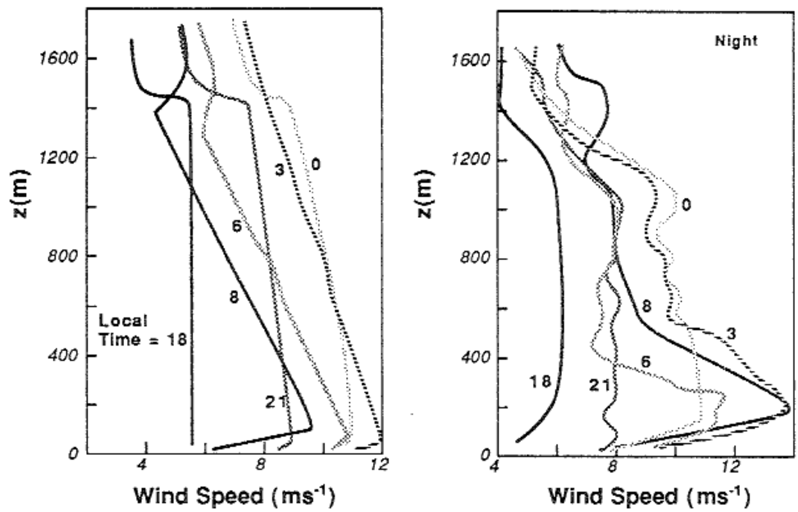

Fig. 3.12 Modeled (left) and observed (right) profiles of mean horizontal wind speed during Night 33-34 of Wangara (André, et al. 1978)

Although there were no latent heating effects during this time period, advection might be important. There were insufficient measurements to calculate the advective contributions to temperature change on this night. Instead, it is left as an exercise for the reader to "back out" the advective contribution, given the data in Fig 3.10.

André, et al. (1978) made a similar analysis of the relative importance of terms in the heat budget equation. Fig 3.11 shows that radiation played a much larger role for the Wangara SBL just after sunset, but that turbulence just above the ground increased in importance later in the night.

Wind speeds (Fig 3.12) observed at night in the Wangara field experiment showed the characteristic nocturnal jet with peak wind speed of about 14 m/s at 200 m above the ground. The simulated profiles shown in the same figure demonstrate the difficulty of forecasting winds at night. Nevertheless, the simulated wind profiles are useful in studying the relative importance of terms in the momentum budget equations (3.5.3 c and d). For this particular case, Fig 3.13 shows that the Coriolis terms were much more important in causing accelerations than were the turbulence effects.

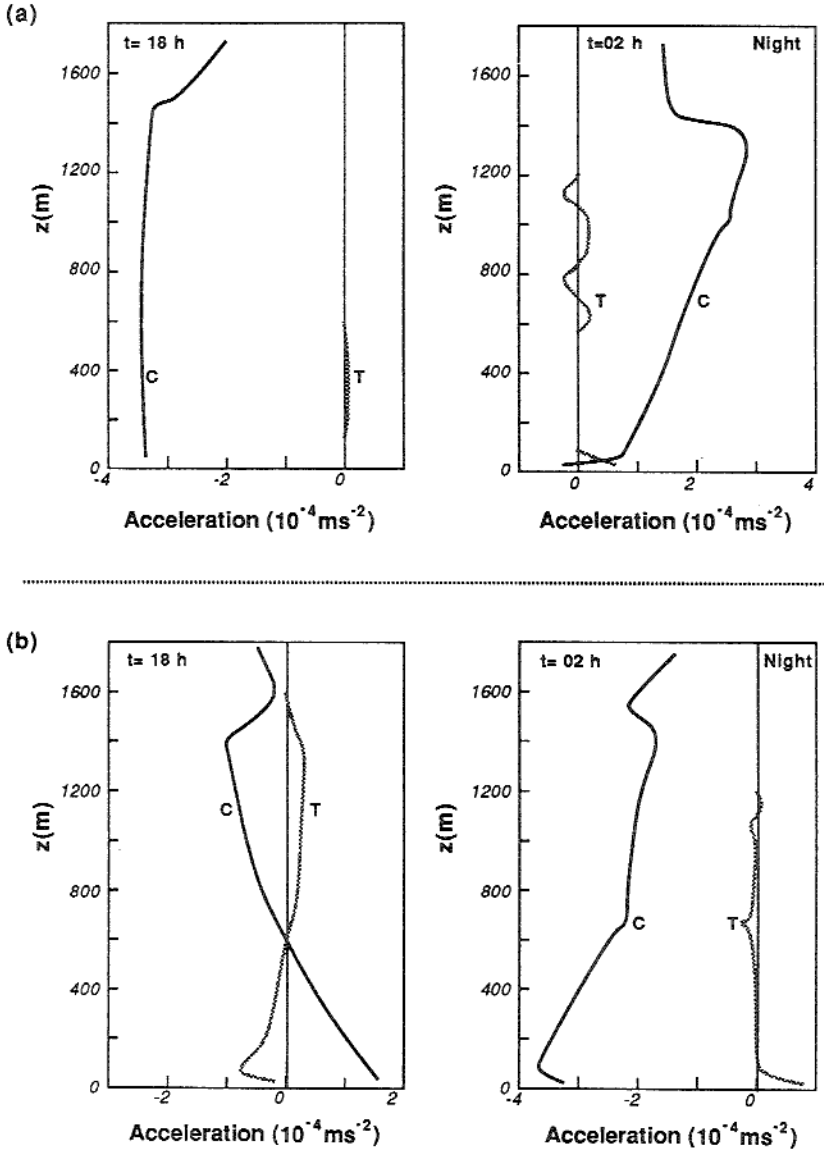

Fig. 3.13 Eastward (a) and northward (b) wind budgets at t=18 h and t=02 h during Night 33-34 of Wangara: C, Coriolis effect; T, turbulent transport (André, et al., 1978).

3.7 References

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3.8 Exercises

1) Suppose that there is an air pollutant called gallacticum that is found within long, narrow spaceships. This pollutant decomposes faster in warmer air than in cooler air. Hence, the conservation equation for gallacticum is

$$\frac{dc}{dt} = - a\, c\, T$$

where c is the concentration of gallacticum, a is a constant, T is the absolute air temperature, and t is time.

Derive the prognostic equation for the mean concentration of gallacticum that applies to turbulent flow within the space ship. You can scale your equations to the space ship by assuming that, within the region of interest, the only mean wind is the forced ventilation current, $\overline{U}$, down the length of the space ship (in the x-direction). There is, however, horizontal homogeneity of mean quantities in the x-direction only. Be sure to put the turbulence term(s) into flux form.

2) Why is an understanding of turbulence necessary for studying and modeling the boundary layer?

3) Expand the following term, and describe its physical meaning.

$$\delta_{ij}\, \overline{U_k}\, \frac{\partial \overline{u_i' u_j'}}{\partial x_k}$$

4) List the steps, assumptions, simplifications and substitutions (in their proper order) used to get the following equation from (3.2.3b). Do NOT do the whole derivation, just list the steps.

$$\frac{\partial \overline{U}}{\partial t} = - f_c (\overline{V_g} - \overline{V}) - \frac{\partial \overline{u'w'}}{\partial z}$$

5) Very briefly define the following, and comment or give examples of their use in micrometeorology:
   a) kinematic heat flux
   b) Reynolds stress
   c) horizontal homogeneity
   d) Boussinesq approximation

6) The forecast equation for mean wind in a turbulent flow is:

$$\underset{A}{\frac{\partial \overline{U_i}}{\partial t}} + \underset{B}{\overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j}} = \underset{C}{- \delta_{i3}\, g} + \underset{D}{f_c\, \varepsilon_{ij3}\, \overline{U_j}} - \underset{E}{\frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i}} + \underset{F}{\nu \frac{\partial^2 \overline{U_i}}{\partial x_j^2}} - \underset{G}{\frac{\partial \overline{u_i' u_j'}}{\partial x_j}}$$

   a) Name each term, and give its physical interpretation.
   b) Starting with the equation above, derive the equation for $\partial \overline{V}/\partial t$, assuming $\overline{U} = 0$.

7) Given the nighttime data of Fig 3.10, estimate the vertical profile of temperature change associated with the advective contribution between 2100 and 2230 CDT.

8) Suppose that the boundary layer warms by 10 °C during a 6 h period. If:

$$\frac{\partial \overline{w'\theta'}}{\partial z} = \frac{\overline{w'\theta'}_{top} - \overline{w'\theta'}_{bottom}}{z_{top} - z_{bottom}}$$

a) then what is the average value of turbulent heat flux at the earth's surface for a 1 km thick BL having no heat flux at its top?

b) If $u_* = 0.2$ m/s, then find θ_*^{SL}.

9) Suppose that:

$\overline{u'w'} = -(u_* + c\,z)^2$, $\overline{v'w'} = 0$ for all z, $\overline{U}_g = 5$ m/s at all heights, $\overline{V}_g = 5$ m/s at all heights, $f_c = 10^{-4}\ s^{-1}$, $u_* = 0.3$ m/s, and $c = 0.001\ s^{-1}$.

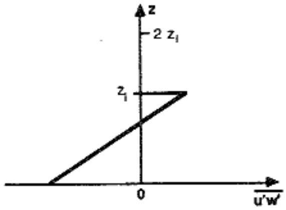
Find the acceleration of the air in the x-direction at a height of 100 m in the BL, assuming that initially $\overline{U} = 4$ m/s and $\overline{V} = 2$ m/s at that height.

10) Given the term $U\,\partial V^2/\partial x$, which represents the advection of mean horizontal v-component of kinetic energy. Expand the variables U and V into mean and turbulent parts, Reynolds average, and simplify as much as possible.

11) Show the steps necessary to put $\overline{u_j'\,\partial T'^2/\partial x_j}$ into flux form.

12) Expand the following Coriolis term $-2\,\varepsilon_{ijk}\,\Omega_j\,U_k$ for the case of i=1, in terms of latitude, velocity, and rotation rate of the earth. Assume that there is no subsidence.

13) Given the profile of momentum flux, $\overline{u'w'}$, sketched here, sketch a mean wind profile between z=0 and z=2z_i that could reasonably occur and be consistant with the momentum flux. Assume a slab-like mixed layer.



14) Let C be the concentration of hockipuculis bacteria in the air. This contagious bacteria, which sweeps across the northern states each winter, is known to increase as ice forms on the lakes. Researchers at the Institute for Sieve Studies have discovered the following conservation equation for hockipuculis in the air:

$$\frac{dC}{dt} = \frac{a\,C}{\overline{\theta}}$$

where "a" is a constant. Find the conservation equation for $\overline{C}$ in a turbulent atmosphere. Assume horizontal homogeneity and no subsidence.

15) Evaluate $H = M\,N$, where $M = \varepsilon_{ijk} \dfrac{\partial \overline{U_j}}{\partial x_i}$ and $N = \delta_{2k}\, \overline{\theta'^2}$.

16) A virulent gas called cyclonide has recently been detected near weathermap display areas. Meteorologists who enhale this gas frequently become euphoric when hurricanes, tornadoes, low-pressure systems, and other cyclones are being displayed on the weather maps.

In your efforts to eliminate this scourge of the meteorological community, you have discovered that cyclonide is neither created nor destroyed but is advected from place to place. If some of this gas escapes into the turbulent boundary layer, you will be asked to forecast its mean concentration. In anticipation of this request, derive the prognostic equation for mean cyclonide concentration. State any assumptions made.

17) Given a kinematic heat flux of 0.2 K m/s at the ground, and a flux of -0.1 K m/s at the top of a 1 km thick mixed layer, calculate the average warming rate of the mixed layer.

18) Expand the following, and eliminate all terms that are zero:

$$\frac{\partial(\varepsilon_{jkl}\, U_i\, U_m\, \delta_{ki}\, \delta_{jl})}{\partial x_m}$$

19) A consortium of personal computer manufacturers has contracted with a local genetic engineering firm to create a new virus call RFV. When humans breath this virus, it causes Ramchip Fever. Symptoms include: an insatiable urge to buy a computer, keyboard finger twitch, memory overflow, a love for mouses, and severe joystick spasms. Parents who breath RFV develop a guilt complex that their offspring will flunk out of school unless they buy a computer.

The concentration, c, of RFV in the air is governed by the following conservation equation:

$$\frac{dc}{dt} = a\, c^2\, T$$

where "a" is a constant and T is absolute temperature.

a) Derive the forecast equation for $\overline{C}$ in turbulent air. Put it into flux form.

b) Scale the answer by assuming horizontal homogeneity and no subsidence.

20) Starting with (3.2.4b), derive a forecast equation for $\overline{q}$ in turbulent flow. That is, derive an equation like (3.4.4b), except for water vapor only. State all assumptions and simplifications used.

21) If a volume of boundary layer air initially contains 2 g/kg of liquid water droplets, and these droplets completely evaporate during 15 minutes, then find $\partial\overline{q}/\partial t$ and $\partial\overline{\theta}/\partial t$ associated with this evaporation. What is the value (with its units) of E, in (3.2.4b) and (3.2.5)?

22) Given typical values for atmospheric air density near sea level, find:

a) $\overline{\rho}g$ in units of (mb/m), and in units of (kPa/m) [useful for converting between pressure and height coordinates].

b) $\overline{\rho}\,C_p$ in units of (mb/ K), and in units of (kPa/K). State all assumptions used.

23) Given typical mean air densities and virtual potential temperatures at sea level in the boundary layer:

a) Find the density fluctuation, ρ', that corresponds to an air parcel with $\theta_v' = +2$ C.

b) Find the vertical acceleration of that air parcel, neglecting pressure and viscous effects.

c) Find the pressure fluctuation, p', if the parcel is restrained from accelerating (neglect viscous effects).

24) What magnitude of V-component geostrophic departure (deviation of the actual wind from its geostrophic value) is necessary to cause the U-component of wind to accelerate 5m/s in one hour? State all assumptions.

25) Use current weather maps (analyses and/or forecasts) to evaluate terms I through IV in (3.3.4) for any one location of interest such as the town you are in. Do it for low level (BL) data for any one variable such as potential temperature or humidity. Compare and discuss the magnitudes of these terms.

26) Given $\overline{u'w'} = -0.3\ m^2\,s^{-2}$, find the value of the Reynolds stress in units of $N{\cdot}m^{-2}$.

27) Look up the value for thermal diffusivity, ν_θ, for air at sea level. Given this value, what curvature in the mean temperature profile would be necessary to cause a warming rate of 5 K/hr? Where, if anywhere, would such curvatures be expected to be found in the boundary layer?

28) Given equations (3.5.3), list all of the necessary initial and boundary conditions necessary to solve those equations for $\overline{\theta}$, $\overline{q}$, $\overline{U}$ and $\overline{V}$.

29) a) Determine the warming rate in the mixed layer, given the soundings of Fig 3.5.

b) Using the heat flux data of Fig 3.2, what percentage of the warming rate from part (a) can be explained by the turbulent flux divergence term?

c) Suggest physical mechanisms to explain the remaining percentages of warming.

30) Same as question (30), except for moisture using Figs 3.6 and 3.2.