4 Prognostic Equations for Turbulent Fluxes and Variances

In the previous chapter, we summarized the equations needed to forecast mean wind, temperature, humidity, and pollutants. The last term in each of equations (3.5.3c) through (3.5.3g) contains a covariance like $\overline{u_j'\theta'}$ or $\overline{u_j'c'}$. In order to use those previous equations, we can either evaluate the covariances experimentally, or we can derive additional equations to forecast the covariances.

In this chapter, prognostic equations are derived for variances and covariances. Variances give us information about turbulence energies and intensities, while covariances describe kinematic turbulent fluxes. While the previous chapter dealt primarily with the mean state, this chapter deals with the turbulent state of the atmosphere.

4.1 Prognostic Equations for the Turbulent Departures

Turbulent departures of variables are the deviations from their respective means; i.e., θ', u', v', w', q', and c'. In theory, prognostic equations for these departures could be used to forecast each individual gust, given accurate initial and boundary conditions for the gust. Unfortunately, the time span over which such a forecast is likely to be accurate is proportional to the lifetime of the eddy itself — on the order of a few seconds for the smallest eddy to about 15 minutes for the larger thermals. For most meteorological applications, such durations are too short to be of direct use. Instead, we will use the prognostic equations derived in this section as an intermediate step towards finding forecast equations for variances and covariances of the variables.

4.1.1 Momentum

Start with the expanded version of the momentum conservation equation (3.4.3a), rewritten here for convenience:

$$\frac{\partial \overline{U_i}}{\partial t} + \frac{\partial u_i'}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} + \overline{U_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{U_i}}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} =$$

$$-\delta_{i3}g + \delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g + f_c\varepsilon_{ij3}\overline{U_j} + f_c\varepsilon_{ij3}u_j' - \left(\frac{1}{\overline{\rho}}\right)\frac{\partial \overline{P}}{\partial x_i} - \left(\frac{1}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 \overline{U_i}}{\partial x_j^2} + \nu\frac{\partial^2 u_i'}{\partial x_j^2}$$

Next, from this equation for the total wind ($\overline{U_i} + u_i'$), subtract the mean part (3.4.3c), also rewritten here:

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3}g + f_c\varepsilon_{ij3}\overline{U_j} - \left(\frac{1}{\overline{\rho}}\right)\frac{\partial \overline{P}}{\partial x_i} + \nu\frac{\partial^2 \overline{U_i}}{\partial x_j^2} - \frac{\partial (\overline{u_i'u_j'})}{\partial x_j}$$

This leaves a prognostic equation for just the turbulent gust, u_i':

$$\frac{\partial u_i'}{\partial t} + \overline{U_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{U_i}}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} =$$

$$+\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g + f_c\varepsilon_{ij3}u_j' - \left(\frac{1}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 u_i'}{\partial x_j^2} + \frac{\partial (\overline{u_i'u_j'})}{\partial x_j} \qquad (4.1.1)$$

4.1.2 Moisture

To simplify future derivations, we will focus on just the vapor portion of the total humidity. For specific humidity of water vapor, start with (3.4.4a), except replace every occurrence of q_T by q, and include the phase change term E (see 3.2.4b). For simplicity, we will assume that body force terms S_q and E are mean terms only. The result is:

$$\frac{\partial \overline{q}}{\partial t} + \frac{\partial q'}{\partial t} + \overline{U_j}\frac{\partial \overline{q}}{\partial x_j} + \overline{U_j}\frac{\partial q'}{\partial x_j} + u_j'\frac{\partial \overline{q}}{\partial x_j} + u_j'\frac{\partial q'}{\partial x_j} =$$

$$\nu_q \frac{\partial^2 \bar{q}}{\partial x_j^2} + \nu_q \frac{\partial^2 q'}{\partial x_j^2} + \frac{(S_q + E)}{\bar{\rho}_{air}} \tag{4.1.2a}$$

Next, subtract the equation for the mean (3.4.4b), again replacing every q_T with q:

$$\frac{\partial \bar{q}}{\partial t} + \overline{U}_j \frac{\partial \bar{q}}{\partial x_j} = \nu_q \frac{\partial^2 \bar{q}}{\partial x_j^2} + \frac{(S_q + E)}{\bar{\rho}_{air}} - \frac{\partial (\overline{u_j' q'})}{\partial x_j} \tag{4.1.2b}$$

leaving a prognostic equation for the perturbation part, q':

$$\frac{\partial q'}{\partial t} + \overline{U}_j \frac{\partial q'}{\partial x_j} + u_j' \frac{\partial \bar{q}}{\partial x_j} + u_j' \frac{\partial q'}{\partial x_j} = \nu_q \frac{\partial^2 q'}{\partial x_j^2} + \frac{\partial (\overline{u_j' q'})}{\partial x_j} \tag{4.1.2c}$$

The reader is invited to derive the equations for the case where S_q and E also have perturbation components.

4.1.3 Heat

Start with (3.4.5a) and subtract (3.4.5b) to leave

$$\frac{\partial \theta'}{\partial t} + \overline{U}_j \frac{\partial \theta'}{\partial x_j} + u_j' \frac{\partial \bar{\theta}}{\partial x_j} + u_j' \frac{\partial \theta'}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta'}{\partial x_j^2} + \frac{\partial (\overline{u_j' \theta'})}{\partial x_j} - \frac{1}{\bar{\rho}\, C_p} \frac{\partial Q_j^{*\prime}}{\partial x_j} \tag{4.1.3}$$

4.1.4 A Scalar Quantity

Start with (3.4.6a) and subtract (3.4.6b) to leave

$$\frac{\partial c'}{\partial t} + \overline{U}_j \frac{\partial c'}{\partial x_j} + u_j' \frac{\partial \overline{C}}{\partial x_j} + u_j' \frac{\partial c'}{\partial x_j} = \nu_c \frac{\partial^2 c'}{\partial x_j^2} + \frac{\partial (\overline{u_j' c'})}{\partial x_j} \tag{4.1.4}$$

4.2 Free Convection Scaling Variables

Before deriving equations for variances and fluxes, we must detour a bit to learn how experimental data is scaled for presentation. We can then show case study examples of data that correspond to the equations we develop.

In Chapter 1, it was stated that turbulence can be produced by buoyant convective processes (i.e., thermals of warm air rising) and by mechanical processes (i.e., wind shear). Sometimes one process dominates. When buoyant convective processes dominate, the boundary layer is said to be in a state of ***free convection***. When mechanical processes dominate, the boundary layer is in a state of ***forced convection***. Free convection occurs over land on clear sunny days with light or calm winds. Forced convection occurs on overcast days with stronger winds. In this section, we will focus on free-convection scales; forced-convection scales have already been introduced in section 2.10.

For the free-convection case, strong solar heating at the surface creates a pronounced diurnal cycle in turbulence and ML depth. In chapter 3, profiles of heat and moisture flux were made nondimensional to remove these diurnal changes. The resulting profiles of heat flux, for example, presented height in terms of a fraction of the total ML depth, and presented flux values as a fraction of the surface flux values.

Such a scheme to remove nonstationary effects can be easily applied to other variables, and is quite useful for studying the relative contributions of the various terms in the variance and flux equations just presented. Some of the appropriate scaling variables for free convection conditions are presented here. Appendix A lists a more complete summary of scaling variables.

Length Scale: Thermals rise until they hit the stable layer capping the ML. As a result, the thermal size scales to z_i. Thermals are the dominant eddy in the convective boundary layer, and all smaller eddies feed on the thermals for energy. Thus, we would expect many turbulent processes to scale to z_i in convective situations.

Velocity Scale: The strong diurnal cycle in solar heating creates a strong heat flux into the air from the earth's surface. The buoyancy associated with this flux fuels the thermals. We can define a ***buoyancy flux*** as $(g/\overline{\theta_v})\,\overline{w'\theta_v'}$.

Although the surface buoyancy flux could be used directly as a scaling variable, it is usually more convenient to generate a velocity scale instead, using the two variables we know to be important in free convection: buoyancy flux at the surface, and z_i. Combining these yields a velocity scale known as the ***free convection scaling velocity***, w_*, also sometimes called the ***convective velocity scale*** for short:

$$w_* = \left[\frac{g\, z_i}{\overline{\theta_v}} \left(\overline{w'\theta_v'}\right)_s \right]^{1/3} \tag{4.2a}$$

This scale appears to work quite well; for example, the magnitude of the vertical velocity fluctuations in thermals is on the same order as w_*. For deep MLs with vigorous heating at the ground, w_* can be on the order of 1 to 2 m/s. Fig 4.1 shows examples of the diurnal variation of w_*.

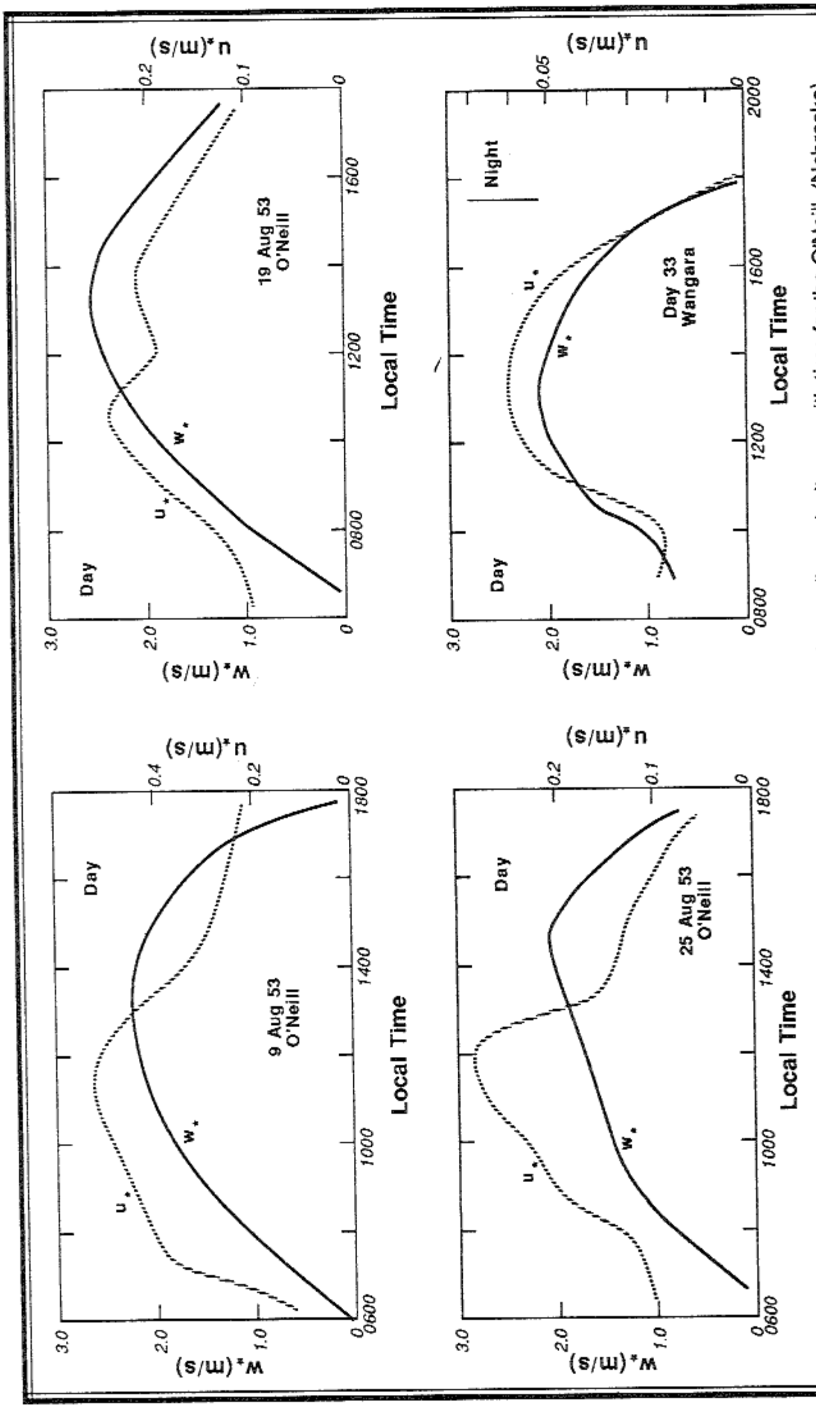


Fig. 4.1 Sample variations of the friction velocity, u_*, and the convective scaling velocity, w_*, with time for the O'Neill, (Nebraska) and Wangara (Australia) field programs.

Time Scale: The velocity and length scales can be combined to give the following free convection time scale, t_*:

$$t_* = \frac{z_i}{w_*} \tag{4.2b}$$

This time scale is on the order of 5 to 15 minutes for many MLs. Observations suggest that this is roughly the time it takes for air in a thermal to cycle once between the bottom and the top of the ML.

Temperature Scale: Using surface heat flux with w_*, we can define a temperature scale for the mixed layer, θ_*^{ML}, by:

$$\theta_*^{ML} = \frac{\left(\overline{w'\theta'}\right)_s}{w_*} \tag{4.2c}$$

This scale is on the order of 0.01 to 0.3 K, which is roughly how much warmer thermals are than their environment.

Humidity Scale: Surface moisture flux and w_* can be combined to define a mixed layer humidity scale, q_*^{ML}:

$$q_*^{ML} = \frac{\left(\overline{w'q'}\right)_s}{w_*} \tag{4.2d}$$

Magnitudes are on the order of 0.01 to 0.5 $g_{water}\,(kg_{air})^{-1}$ and scale well to moisture excesses within thermals.

With these convective scales in mind, we can return to the equation derivations.

4.3 Prognostic Equations for Variances

4.3.1 Momentum Variance

Basic Derivation. Start with (4.1.1) and multiply by $2u_i'$:

$$2u_i'\frac{\partial u_i'}{\partial t} + 2\overline{U}_j u_i'\frac{\partial u_i'}{\partial x_j} + 2u_i'u_j'\frac{\partial \overline{U}_i}{\partial x_j} + 2u_i'u_j'\frac{\partial u_i'}{\partial x_j} =$$

$$+ 2\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta}_v}\right)g + 2f_c\,\varepsilon_{ij3}u_i'u_j' - 2\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i} + 2\nu\, u_i'\frac{\partial^2 u_i'}{\partial x_j^2} + 2u_i'\frac{\partial\left(\overline{u_i'u_j'}\right)}{\partial x_j}$$

Next, use the product rule of calculus to convert terms like $2u_i'\partial u_i'/\partial t$ into $\partial(u_i')^2/\partial t$:

$$\frac{\partial u_i'^2}{\partial t} + \overline{U}_j \frac{\partial u_i'^2}{\partial x_j} + 2u_i'u_j' \frac{\partial \overline{U}_i}{\partial x_j} + u_j' \frac{\partial u_i'^2}{\partial x_j} =$$

$$+ 2\,\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta}_v}\right)g + 2f_c\,\varepsilon_{ij3}u_i'u_j' - 2\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i} + 2\nu u_i' \frac{\partial^2 u_i'}{\partial x_j^2} + 2u_i' \frac{\partial\, (\overline{u_i'u_j'})}{\partial x_j}$$

For step three, average the whole equation and apply Reynolds averaging rules:

$$\overline{\frac{\partial u_i'^2}{\partial t}} + \overline{\overline{U}_j \frac{\partial u_i'^2}{\partial x_j}} + \overline{2u_i'u_j' \frac{\partial \overline{U}_i}{\partial x_j}} + \overline{u_j' \frac{\partial u_i'^2}{\partial x_j}} =$$

$$+ \overline{2\,\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta}_v}\right)g} + \overline{2f_c\,\varepsilon_{ij3}u_i'u_j'} - \overline{2\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} + \overline{2\nu u_i' \frac{\partial^2 u_i'}{\partial x_j^2}} + \overline{2u_i' \frac{\partial\, (\overline{u_i'u_j'})}{\partial x_j}}$$

where the last term is zero because $\overline{u_i'} = 0$. If we multiply the turbulent continuity equation by $u_i'^2$ and Reynolds average to get $\overline{u_i'^2\, \partial u_j'/\partial x_j} = 0$, then we can add this equation to the equation above to put the last term before the equal sign into ***flux form***: $\partial(\overline{u_j'\, u_i'^2})/\partial x_j$. This leaves:

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{U}_j \frac{\partial \overline{u_i'^2}}{\partial x_j} + 2\overline{u_i'u_j'} \frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial\, (\overline{u_j'u_i'^2}\,)}{\partial x_j} =$$

$$+ 2\,\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\overline{\theta}_v}\right)}g + 2f_c\,\varepsilon_{ij3}\overline{u_i'u_j'} - 2\overline{\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} + 2\nu \overline{u_i' \frac{\partial^2 u_i'}{\partial x_j^2}} \qquad (4.3.1a)$$

This general form of the prognostic equation for the variance of wind speed, $\overline{u_i'^2}$, is usually simplified further before being used for boundary layer flows.

Dissipation. Consider a term of the form $\partial^2(\overline{u_i'^2})/\partial x_j^2$. Using simple rules of calculus, we can rewrite it as:

$$\frac{\partial^2(\overline{u_i'^2})}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left[\frac{\partial(\overline{u_i'^2})}{\partial x_j}\right] = \frac{\partial}{\partial x_j}\left[\overline{2u_i'\frac{\partial u_i'}{\partial x_j}}\right] = 2\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + \overline{2u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} =$$

$$2\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{2u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}$$

If we multiply the last term above by ν, then it would be identical to the last term in (4.3.1a). Thus, we can write the last term in (4.3.1a) as

$$2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} = \nu\frac{\partial^2(\overline{u_i'^2})}{\partial x_j^2} - 2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \qquad (4.3.1b)$$

The first term on the right, which physically represents the molecular diffusion of velocity variance, contains the curvature of a variance. The variance changes fairly smoothly with distance within the boundary layer, its curvature being on the order of 10^{-6} s^{-2} in the ML to 10^{-2} s^{-2} in the SL. When multiplied by ν, the first term ranges in magnitude between 10^{-11} and 10^{-7} $m^2 s^{-3}$.

The last term on the right can be much larger. For example, if the eddy velocity changes by only 0.1 m/s across a very small size eddy (for example, 1 cm in diameter), then the instantaneous shear across that eddy is 10 s^{-1}. For smaller size eddies, the shear is larger. When this value is squared, averaged, and multiplied by 2ν, the magnitudes observed in the turbulent boundary layer range between about 10^{-6} and 10^{-2} $m^2 s^{-3}$. Typical values in the ML are on the order of 10^{-4} to 10^{-3} $m^2 s^{-3}$, while in the surface layer, values on the order of 10^{-2} $m^2 s^{-3}$ can be found. Thus, we can neglect the first term on the right and use:

$$2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} \cong -2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \qquad (4.3.1c)$$

The ***viscous dissipation***, ε, is defined as:

$$\varepsilon = +\nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \qquad (4.3.1d)$$

It is obvious that this term is always positive, because it is a squared quantity. Therefore, when used in (4.3.1a) with the negative sign as required by (4.3.1c), it is always causing a decrease in the variance with time. That is, ***it is always a loss term***. In addition, it becomes larger in magnitude as the eddy size becomes smaller. For these small eddies, the eddy motions are rapidly damped by viscosity and irreversibly converted into heat. [This heating rate is so small, however, that it has been neglected in the heat conservation equation (3.4.5b).]

Pressure Perturbations. Using the product rule of calculus again, the pressure term $-2\,\overline{(u_i'/\bar{\rho})\,\partial p'/\partial x_i}$ in (4.3.1a) can be rewritten as

$$-2\overline{\left(\frac{u_i'}{\bar{\rho}}\right)\frac{\partial p'}{\partial x_i}} = -\left(\frac{2}{\bar{\rho}}\right)\frac{\partial\,(\overline{u_i'p'})}{\partial x_i} + 2\overline{\left(\frac{p'}{\bar{\rho}}\right)\left[\frac{\partial u_i'}{\partial x_i}\right]}$$

The last term is called the ***pressure redistribution term***. The factor in square brackets consists of the sum of three terms: $\partial u'/\partial x$, $\partial v'/\partial y$, and $\partial w'/\partial z$. These terms sum to zero because of the turbulence continuity equation (3.4.2c); hence, the last term in the equation above does not change the total variance (by total variance we mean the sum of all three variance components). But it does tend to take energy out of the components having the most energy and put it into components with less energy. Thus it makes the turbulence more isotropic, and is also known as the ***return-to-isotropy term***.

Terms like $\partial u'/\partial x$ are larger for the smaller size eddies. Thus, we would expect that smaller size eddies are more isotropic than larger ones. As we shall see later, this is indeed the case in the boundary layer.

The end result of this analysis is that:

$$-2\overline{\left(\frac{u_i'}{\bar{\rho}}\right)\frac{\partial p'}{\partial x_i}} \cong -\left(\frac{2}{\bar{\rho}}\right)\frac{\partial\,(\overline{u_i'p'})}{\partial x_i} \qquad (4.3.1e)$$

Coriolis Term. The Coriolis term $2f_c\varepsilon_{ij3}\,\overline{u_i'u_j'}$ is identically zero for velocity variances, as can be seen by performing the sums implied by the repeated indices:

$$2f_c\varepsilon_{ij3}\,\overline{u_i'u_j'} = 2f_c\varepsilon_{213}\,\overline{u_2'u_1'} + 2f_c\varepsilon_{123}\,\overline{u_1'u_2'}$$

$$= -2f_c\overline{u_2'u_1'} + 2f_c\overline{u_1'u_2'}$$

$$= 0 \qquad (4.3.1f)$$

because $\overline{u_1'u_2'} = \overline{u_2'u_1'}$ (see section 2.9.2). Many of the terms in the above sum were not written out because the alternating unit tensor forced them to zero.

Physically, this means that Coriolis force can not generate turbulent kinetic energy. Kinetic energy enters the picture because the variance $\overline{u_i'^2}$ is nothing more than twice the turbulence kinetic energy per unit mass. The Coriolis term merely redistributes energy from one horizontal direction to another. Furthermore, the magnitude of the redistribution term $2f_c\,\overline{u_1'u_2'}$ is about three orders of magnitude smaller than the other terms in (4.3.1a). For that reason, the Coriolis terms are usually neglected in the turbulence variance and covariance equations, even for the cases where they are not identically zero.

Simplified Velocity Variance Budget Equations. Inserting the simplifications of the previous subsections in equation (4.3.1a) and rearranging the terms gives:

$$\underset{\text{I}}{\frac{\partial\overline{u_i'^2}}{\partial t}} + \underset{\text{II}}{\overline{U}_j\frac{\partial\overline{u_i'^2}}{\partial x_j}} = \underset{\text{III}}{+2\,\delta_{i3}\frac{g\,(\overline{u_i'\theta_v'})}{\overline{\theta}_v}} \; \underset{\text{IV}}{-\,2\overline{u_i'u_j'}\frac{\partial\overline{U}_i}{\partial x_j}} \; \underset{\text{V}}{-\,\frac{\partial\,(\overline{u_j'u_i'^2})}{\partial x_j}} \; \underset{\text{VI}}{-\,\frac{2}{\overline{\rho}}\frac{\partial\,(\overline{u_i'p'})}{\partial x_i}} \; \underset{\text{VII}}{-\,2\varepsilon} \qquad (4.3.1g)$$

Term I represents local storage of variance.
Term II describes the advection of variance by the mean wind.
Term III is a production or loss term, depending on whether the buoyancy flux $\overline{w'\theta_v'}$ is positive (e.g., daytime over land) or negative (e.g., night over land).

Term IV is a production term. The momentum flux $\overline{u_i'u_j'}$ is usually negative in the boundary layer because the momentum of the wind is lost downward to the ground; thus, it results in a positive contribution to variance when multiplied by a negative sign.

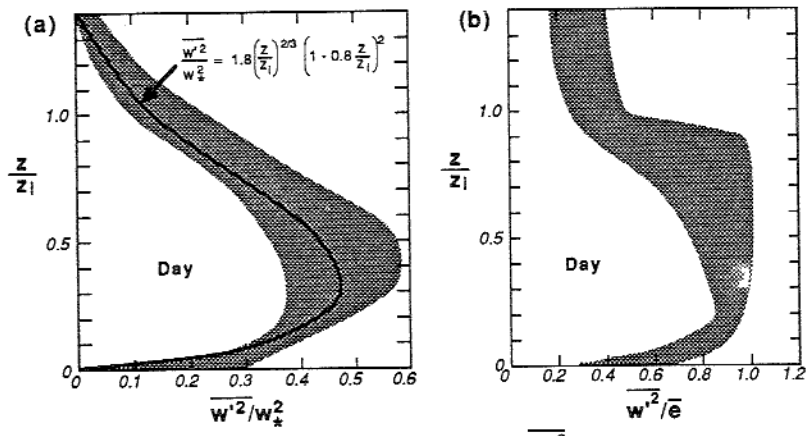


Fig. 4.2 (a) Variation of vertical velocity variance, $\overline{w'^2}$, with height, z during daytime. Height is normalized with respect to the average mixed layer depth, z_i, while variance is normalized by the convective velocity scale, w_*. Range of measured and modeled values are shaded, with a simple empirical approximation given by the solid line. (b) Range of the ratio of the vertical velocity variance to the eddy kinetic energy, $\bar{e}$. Based on data from Deardorff (1974), Lenschow, et al. (1980), André et al. (1978), Therry and Lacarrere (1983), and Smedman and Högström (1983).

Term V is a turbulent transport term. It describes how variance $\overline{u_i'^2}$ is moved around by the turbulent eddies u_j'.

Term VI describes how variance is redistributed by pressure perturbations. It is often associated with oscillations in the air (i.e., ***buoyancy or gravity waves***).

Term VII represents the viscous dissipation of velocity variance.

We can also examine the prognostic equations for each individual component of velocity variance if we relax slightly the summation requirement associated with repeated indices. For example, in the above equation, we could let i=2 to write the forecast equation for $\overline{v'^2}$. Any other repeated indices, such as j, continue to imply a sum. When we perform such a split, remembering to reinsert the return-to-isotropy terms (because for any one component, it is nonzero), we find:

$$\frac{\partial \overline{u'^2}}{\partial t} + \overline{U}_j \frac{\partial \overline{u'^2}}{\partial x_j} = -2\overline{u'u_j'}\frac{\partial \overline{U}}{\partial x_j} - \frac{\partial (\overline{u_j'u'^2})}{\partial x_j} - \frac{2}{\overline{\rho}}\frac{\partial(\overline{u'p'})}{\partial x} + \frac{2\overline{p'}}{\overline{\rho}}\frac{\partial u'}{\partial x} - 2\nu\overline{\left(\frac{\partial u'}{\partial x_j}\right)^2} \tag{4.3.1h}$$

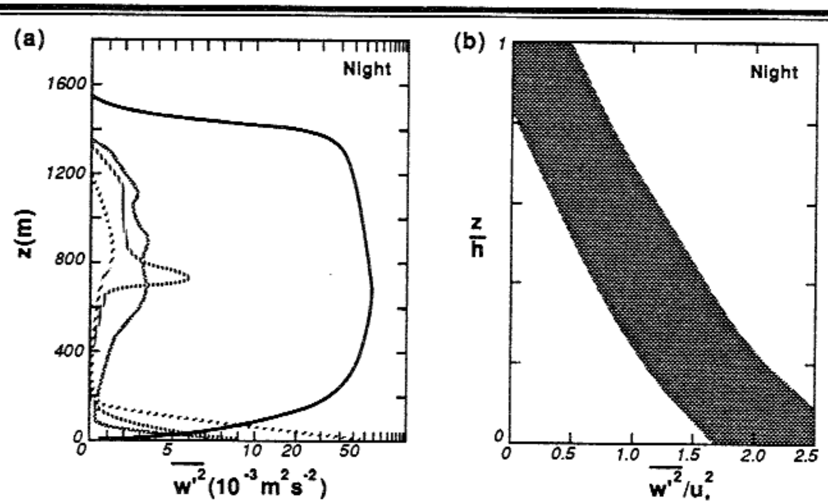


Fig. 4.3 (a) Modeled profiles of vertical velocity variance during Night 33-34 of Wangara: ——, t=18 h; ——, t=21 h; ·······, t=03 h; ,,,,,,, t=07 h. Abscissa changes from linear to logarithmic at 10. (b) Range of Vertical velocity variance, $\overline{w'^2}$, normalized by friction velocity, u_*, plotted as a function of height, z, normalized by a measure of stable boundary layer depth, h. Based on data from André et al. (1978) and Caughey, et al. (1979).

$$\frac{\partial \overline{v'^2}}{\partial t} + \overline{U}_j \frac{\partial \overline{v'^2}}{\partial x_j} = -2\overline{v'u_j'}\frac{\partial \overline{V}}{\partial x_j} - \frac{\partial (\overline{u_j'v'^2})}{\partial x_j} - \frac{2}{\overline{\rho}}\frac{\partial(\overline{v'p'})}{\partial y} + \overline{\frac{2p'}{\overline{\rho}}\frac{\partial v'}{\partial y}} - 2\nu\overline{\left(\frac{\partial v'}{\partial x_j}\right)^2} \tag{4.3.1i}$$

$$\underset{\text{I}}{\frac{\partial \overline{w'^2}}{\partial t}} + \underset{\text{II}}{\overline{U}_j \frac{\partial \overline{w'^2}}{\partial x_j}} = \underset{\text{III}}{\frac{2g(\overline{w'\theta_v'})}{\overline{\theta_v}}} \underset{\text{IV}}{- 2\overline{w'u_j'}\frac{\partial \overline{W}}{\partial x_j}} \underset{\text{V}}{- \frac{\partial(\overline{u_j'w'^2})}{\partial x_j}} \underset{\text{VI}}{- \frac{2}{\overline{\rho}}\frac{\partial(\overline{w'p'})}{\partial z}} \underset{\text{VIII}}{+ \overline{\frac{2p'}{\overline{\rho}}\frac{\partial w'}{\partial z}}} \underset{\text{VII}}{- 2\nu\overline{\left(\frac{\partial w'}{\partial x_j}\right)^2}} \tag{4.3.1.j}$$

Terms I through VII have the same meaning as before. Term VIII represents pressure redistribution, which is associated with the return-to-isotropy term.

Case Study Examples. *Budget study* is the name given to an evaluation of the contributions of each term in prognostic equations such as the ones just derived. Some terms are very difficult to measure in field experiments, which is why computer simulation efforts are made. In the budget studies that follow, field data and numerical simulations are combined, and the range of values is indicated. In most cases, field measurements have significantly more scatter than the simulations.

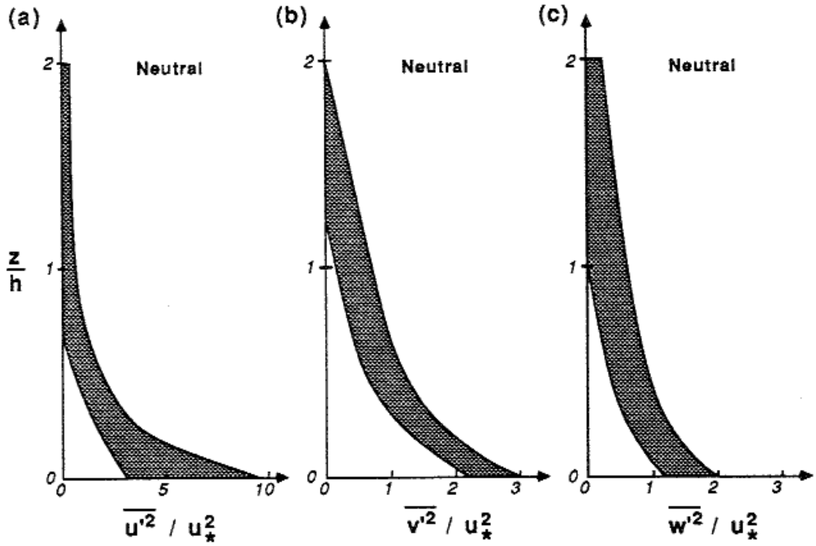


Fig. 4.4 Normalized velocity variance verses height in statically neutral conditions, where h (≈2km) is the height where $\overline{v}$ is zero. Based on a large-eddy simulation by Mason and Thomson (1987) using U_g=10m/s, V_g=0, and u_*= 0.4 m/s.

Fig 4.2 shows that vertical velocity variance during the daytime is small near the surface, increases to a maximum about a third of the distance from the ground to the top of the ML, and then decreases with height. This is related to the vertical acceleration experienced by thermals during their initial rise, which is reduced by dilution with environmental air, by drag, and by the warming and stabilizing of the environment near the top of the ML. In cloud-free conditions with light winds, glider pilots and birds would expect to find the maximum lift at $z/z_i = 0.3$.

At night, turbulence rapidly decreases over the residual layer, leaving a much thinner layer of turbulent air near the ground, as is shown in Fig 4.3. The depth of this turbulent SBL is often relatively small (h ≅ 200 m). In statically neutral conditions the variances also decrease with height from large values at the surface, as shown in Fig 4.4 (Mason and Thompson, 1987); however, the depth scale is much larger (h ≅ 2 km).

Fig 4.5 shows that the horizontal components are often largest near the ground during the day, associated with the strong wind shears in the surface layer. The horizontal variance is roughly constant throughout the ML, but decreases with height above the ML top. At night, the horizontal variance decreases rapidly with height to near zero at the top of the SBL (Fig 4.6). This shape is similar to that of the vertical velocity variance.

The budget term discussion for velocity variance will be deferred to Chapter 5 because of the close association of velocity variance with turbulent kinetic energy.

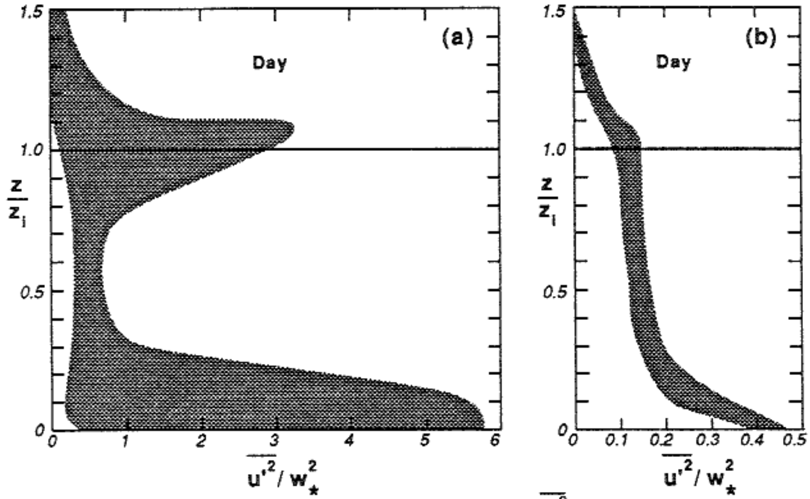


Fig. 4.5 (a) Range of horizontal velocity variance, $\overline{u'^2}$, normalized by the convective velocity scale, w_*^2, as a function of dimensionless height z/z_i, for typical conditions with combined convection and wind shear. (b) Idealized range for free convection with no mean shear. Based on data from Smedman and Hogstrom (1983), Deardorff (1974), André et al. (1978), and Lenschow et al. (1980).

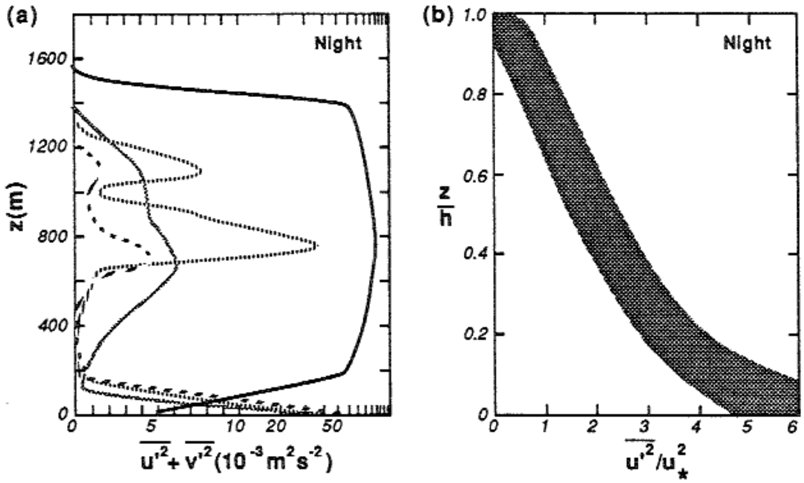


Fig. 4.6 (a) Modeled profiles of horizontal eddy kinetic energy during Night 33-34 of Wangara: ▬▬, t=18 h; ═══, t=21 h; ┈┈┈, t=03 h; - - -, t=07 h. Abscissa changes from linear to logarithmic at 10. (b) Range of horizontal velocity variance, $\overline{u'^2}$, normalized by friction velocity, u_*, plotted as a function of height, z, normalized by a measure of stable boundary layer depth, h. Based on data from André, et al. (1978) and Caughey, et al. (1979).

4.3.2 Moisture Variance

Budget Equation. In the following development, only the vapor part of the specific humidity will be used, although similar derivations could be performed for the nonvapor part too. Start with (4.1.2c), multiply by 2q', and use the product rule of calculus to convert terms like $2q' \partial q'/\partial t$ into terms like $\partial(q'^2)/\partial t$, to yield:

$$\frac{\partial q'^2}{\partial t} + \overline{U}_j \frac{\partial q'^2}{\partial x_j} + 2q'u_j' \frac{\partial \overline{q}}{\partial x_j} + u_j' \frac{\partial q'^2}{\partial x_j} = 2q'\nu_q \frac{\partial^2 q'}{\partial x_j^2} + 2q' \frac{\partial (\overline{u_j'q'})}{\partial x_j}$$

Next, average and apply Reynolds averaging rules:

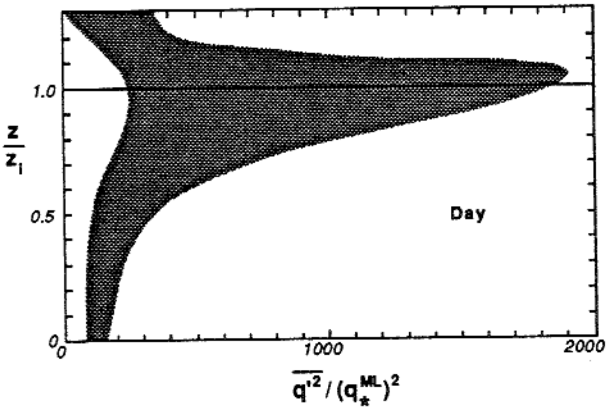
$$\frac{\partial \overline{q'^2}}{\partial t} + \overline{U}_j \frac{\partial \overline{q'^2}}{\partial x_j} + 2\overline{q'u_j'} \frac{\partial \overline{q}}{\partial x_j} + \overline{u_j' \frac{\partial q'^2}{\partial x_j}} = \overline{2q'\nu_q \frac{\partial^2 q'}{\partial x_j^2}}$$

To change this into flux form, add the averaged turbulent continuity equation multiplied by q'^2 (i.e., add $\overline{q'^2 \partial u_j'/\partial x_j} = 0$), and rearrange slightly:

$$\frac{\partial \overline{q'^2}}{\partial t} + \overline{U}_j \frac{\partial \overline{q'^2}}{\partial x_j} = -2\overline{q'u_j'} \frac{\partial \overline{q}}{\partial x_j} - \frac{\partial (\overline{u_j'q'^2})}{\partial x_j} + 2\nu_q \overline{q' \frac{\partial^2 q'}{\partial x_j^2}}$$

As was done for momentum, the last term is split into two parts, one of which (the molecular diffusion of specific humidity variance) is small enough to be neglected. The

Fig. 4.7 Modeled vertical profiles of dimensionless specific-humidity variance for Day 33, Wangara. Based on data from Deardorff (1974) and Lenschow, et al., (1980).



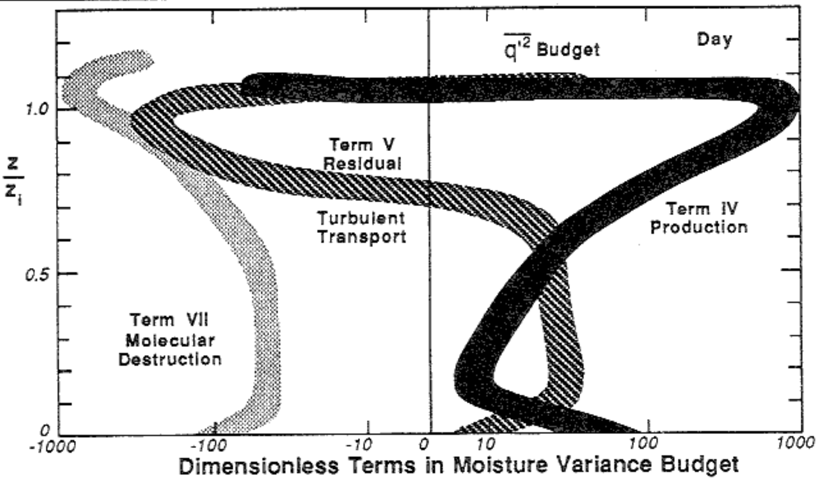


Fig. 4.8 Modeled vertical profiles of terms in the specific-humidity variance equation [see equation (4.3.2)] made dimensionless by dividing by $w_*(q_*^{ML})^2 / z_i$, for Wangara, Day 33, hour 14.1. Abscissa changes from linear to logarithmic at ± 10. $w_* = 2.04$ m/s, $q_* = 1.3 \times 10^{-5}$ g/g, and $z_i = 1305$ m. The thickness of the curves is meant to suggest some uncertainty in the precise values. Based on data from Deardorff (1974) and Lenschow, et al. (1980).

remaining part is defined as twice the molecular dissipation term, ε_q, by analogy with momentum:

$$\varepsilon_q = \nu_q \overline{\left(\frac{\partial q'}{\partial x_j}\right)^2}$$

Thus, the prognostic equation for specific humidity variance is

$$\underset{\text{I}}{\frac{\partial \overline{q'^2}}{\partial t}} + \underset{\text{II}}{\overline{U}_j \frac{\partial \overline{q'^2}}{\partial x_j}} = \underset{\text{IV}}{- 2\overline{q'u_j'} \frac{\partial \overline{q}}{\partial x_j}} - \underset{\text{V}}{\frac{\partial (\overline{u_j'q'^2})}{\partial x_j}} - \underset{\text{VII}}{2\varepsilon_q} \qquad (4.3.2)$$

Term I represents local storage of humidity variance
Term II describes the advection of humidity variance by the mean wind
Term IV is a production term, associated with turbulent motions occurring within a mean moisture gradient
Term V represents the turbulent transport of humidity variance
Term VII is the molecular dissipation.

Case Study Examples. Fig 4.7 shows that humidity variance is small near the ground, because thermals have nearly the same humidity as their environment. At the top of the ML, however, drier air from aloft is being entrained down between the moist thermals, creating large humidity variances. Part of this variance might be associated with the excitation of gravity/buoyancy waves by the penetrative convection.

Fig 4.8 shows production terms balancing loss terms in the budget, assuming a steady state situation where storage and mean advection are neglected. Notice that the transport terms (found as a residual) are positive in the bottom half of the ML, but are negative in the top half. The integrated effects of these terms are zero. Such is the case for most transport terms — they merely move moisture variance from one part of the ML (where there is excess production) to another part (where there is excess dissipation), leaving zero net effect when averaged over the whole ML.

4.3.3 Heat (Potential Temperature Variance)

Budget Equations. As was done with the moisture equation, start with (4.1.3), multiply by 2θ', use the product rule of calculus, Reynolds average, put into flux form, neglect molecular diffusion but retain the molecular dissipation, and rearrange to yield:

$$\frac{\partial \overline{\theta'^2}}{\partial t} + \overline{U}_j \frac{\partial \overline{\theta'^2}}{\partial x_j} = -2\overline{\theta' u_j'}\frac{\partial \overline{\theta}}{\partial x_j} - \frac{\partial(\overline{u_j'\theta'^2})}{\partial x_j} - 2\varepsilon_\theta - \left(\frac{2}{\overline{\rho} C_p}\right)\overline{\theta' \frac{\partial Q_j^{*\prime}}{\partial x_j}} \qquad (4.3.3)$$

I II IV V VII VIII

The terms above have physical representations analogous to those in (4.3.2). Term VIII is the radiation destruction term (sometimes given the symbol ε_R). It is difficult to

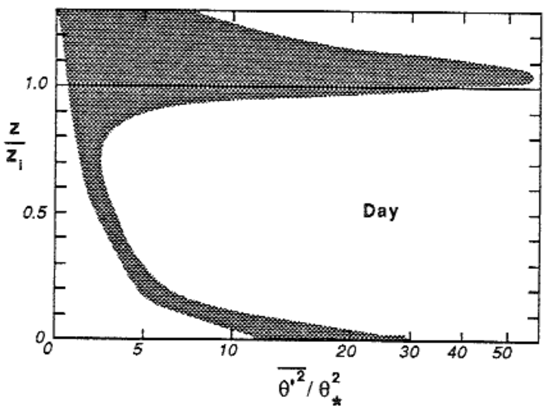


Fig. 4.9 Modeled vertical profiles of dimensionless virtual potential temperature variance for Day 33, Wangara. Abscissa changes from linear to logarithmic at 10. Based on data from Deardorff (1974) André, et al.,(1978), Lenschow, et al. (1980), and Smedman and Högström (1983).

measure this term directly, but sometimes it is modeled as $\varepsilon_R \cong (0.036\ \text{m/s})\cdot\varepsilon\,\overline{\theta'^2}\,/\,\bar{e}^{\,3/2}$, where ε_R is about 1% to 10% of ε_θ (Coantic and Simonin, 1984).

Case Study Examples. The temperature variance at the top of the ML (Fig 4.9) is similar to humidity variance, because of the contrast between warmer entrained air and the cooler overshooting thermals. Gravity waves may also contribute to the variance. There is a greater difference near the bottom of the ML, however, because warm thermals in a cooler environment enhance the magnitude of the variance there. At night, Fig 4.10 shows that the largest temperature fluctuations are near the ground in the NBL, with weaker, sporadic turbulence in the RL aloft.

Fig 4.11 shows the contributions to the heat budget during daytime, again neglecting storage and advection. The radiation destruction term is small, but definitely nonzero. The dissipation is largest near the ground, as is the turbulent transport of temperature variance. Fig 4.12 shows the corresponding budget terms at night.

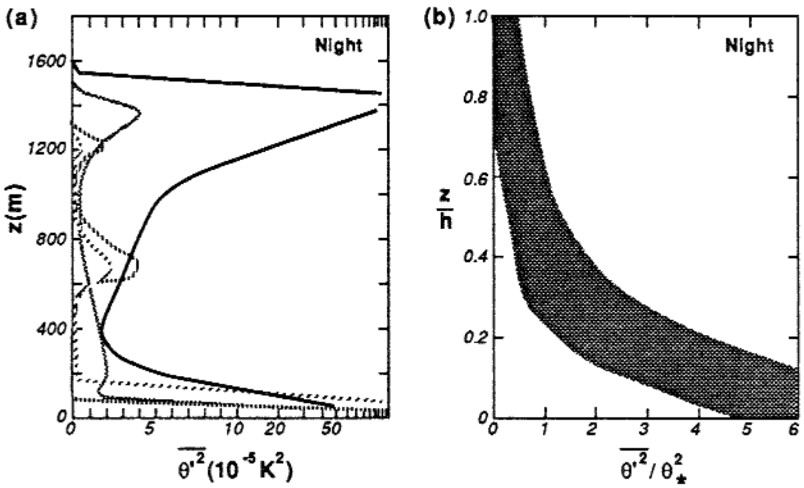


Fig. 4.10 (a) Modeled profiles of virtual potential temperature variance during Night 33-34 of Wangara: symbols as in Fig. 4.3(a). Abscissa changes from linear to logarithmic at 10. (b) Range of temperature variance, $\overline{\theta'^2}$ normalized by surface layer temperature scale, θ_*^{SL}, plotted as a function of height normalized by boundary layer depth. Based on data from André, et al. (1978) and Caughey, et al. (1979).

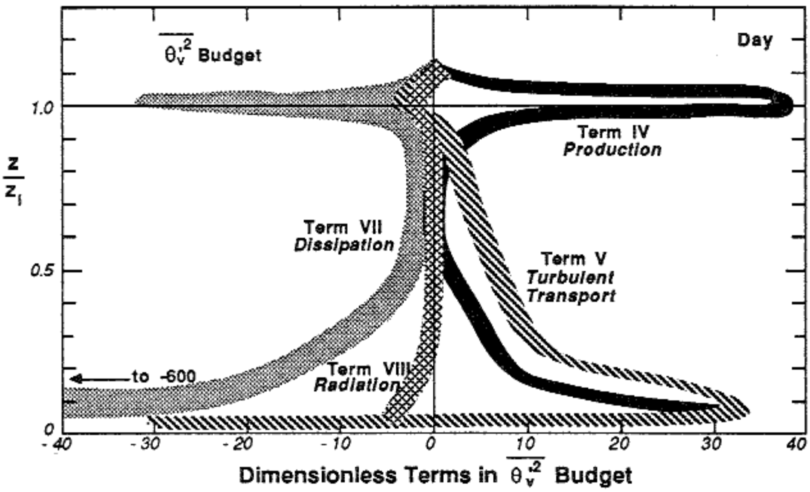


Fig. 4.11 Modeled vertical profiles of terms in the virtual potential temperature variance budget equation made dimensionless by dividing by $w_*(\theta_*^{ML})^2 / z_i$. For Day 33, Wangara, hour 14.1. Thickness of lines denotes uncertainty or variability in the precisevalues.Based on data from Deardorff (1974), Lenschow et al. (1980), Andre et al. (1978), Caughey and Palmer (1979) and Zhou, et al. (1985).

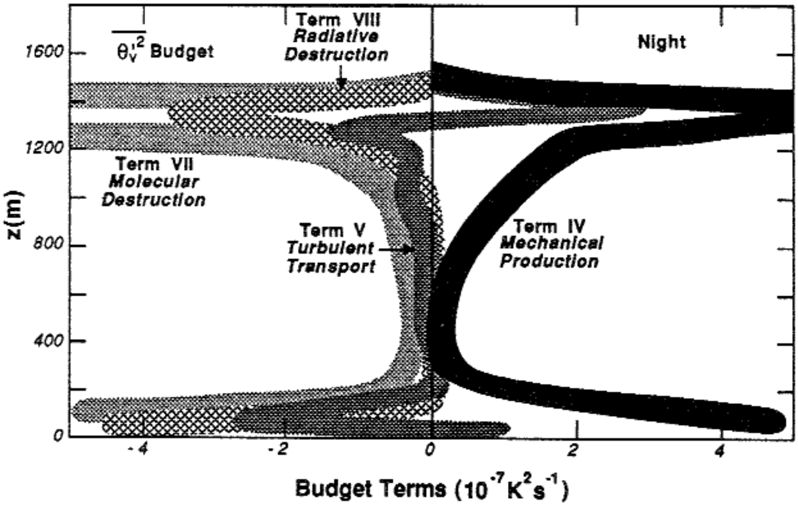


Fig. 4.12 Modeled budgets of virtual potential temperature variance at t=18 h during Night 33-34, Wangara. After André et al. (1978).

4.3.4 A Scalar Quantity (Tracer Concentration Variance)

Analogous with the moisture equation, start with (4.1.4), multiply by 2c', use the product rule of calculus, Reynolds average, put into flux form, neglect molecular diffusion but retain the molecular dissipation, and rearrange to yield:

$$\underset{\text{I}}{\frac{\partial \overline{c'^2}}{\partial t}} + \underset{\text{II}}{\overline{U}_j \frac{\partial \overline{c'^2}}{\partial x_j}} = \underset{\text{IV}}{-2\overline{c'u_j'}\frac{\partial \overline{C}}{\partial x_j}} - \underset{\text{V}}{\frac{\partial (\overline{u_j'c'^2})}{\partial x_j}} - \underset{\text{VII}}{2\varepsilon_c} \qquad (4.3.4)$$

The terms above have physical representations analogous to those in (4.3.2).

No case study examples are shown for tracer variances because they vary so widely from constituent to constituent.

4.4 Prognostic Equations for Turbulent Fluxes

The equations of section 3.5.3 contain divergence terms of turbulent fluxes (e.g., $\overline{u_i'u_j'}$, $\overline{u_j'\theta'}$, $\overline{u_j'q'}$, and $\overline{u_j'c'}$). These fluxes are unknowns in equations (3.5.3). If prognostic equations for the fluxes can be found, one hopes that there will be as many equations as unknowns, allowing determination of the boundary layer wind and turbulence state. In this section, we will derive equations for the unknown fluxes; unfortunately, these new equations will contain additional new unknowns.

4.4.1 Momentum Flux

Budget Equations. Two perturbation equations are combined to produce flux equations. To obtain the first equation, start with (4.1.1), multiply it by u_k', and Reynolds average:

$$\overline{u_k'\frac{\partial u_i'}{\partial t}} + \overline{u_k'\overline{U}_j\frac{\partial u_i'}{\partial x_j}} + \overline{u_k'u_j'\frac{\partial \overline{U}_i}{\partial x_j}} + \overline{u_k'u_j'\frac{\partial u_i'}{\partial x_j}} =$$

$$+ \delta_{i3}\, \overline{u_k'\left(\frac{\theta_v'}{\overline{\theta}_v}\right)} g + f_c\varepsilon_{ij3}\,\overline{u_k'u_j'} - \overline{\left(\frac{u_k'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} + \nu\, \overline{u_k'\frac{\partial^2 u_i'}{\partial x_j^2}}$$

For the second equation, interchange the i and k indices (i.e., replace each occurrence of i with k, and each occurrence of k with i). Such an interchange will not change the meaning of the equation, because summed terms will continue to be summed, and unsummed terms will continue to represent the three components. The result is:

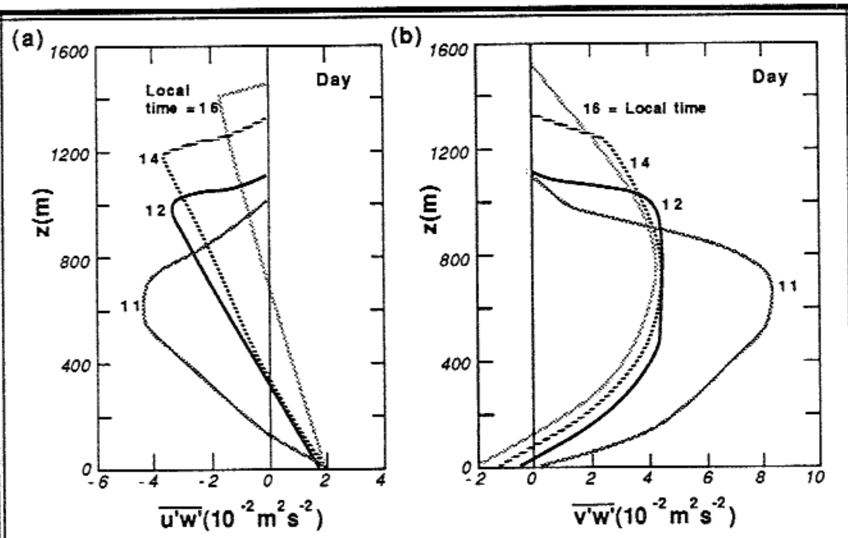


Fig. 4.13 (a) Modeled profiles of vertical turbulent flux of u-wind component during Day 33, Wangara. (b) Modeled profiles of vertical turbulent flux of northward wind component during Day 33, Wangara. After André et al., (1978).

$$\overline{u_i' \frac{\partial u_k}{\partial t}} + \overline{u_i' \overline{U}_j \frac{\partial u_k'}{\partial x_j}} + \overline{u_i' u_j' \frac{\partial \overline{U}_k}{\partial x_j}} + \overline{u_i' u_j' \frac{\partial u_k'}{\partial x_j}} =$$

$$+ \delta_{k3}\, \overline{u_i' \left(\frac{\theta_v'}{\overline{\theta}_v}\right)} + f_c \varepsilon_{kj3}\, \overline{u_i' u_j'} - \overline{\left(\frac{u_i'}{\overline{\rho}}\right) \frac{\partial p'}{\partial x_k}} + \nu\, \overline{u_i' \frac{\partial^2 u_k'}{\partial x_j^2}}$$

Next, add these two equations together, and use the product rule of calculus to produce combinations like $\overline{u_i' \partial u_k'/\partial t} + \overline{u_k' \partial u_i'/\partial t} = \partial(\overline{u_i' u_k'})/\partial t$:

$$\frac{\partial \overline{u_i' u_k'}}{\partial t} + \frac{\overline{U}_j \partial \overline{u_i' u_k'}}{\partial x_j} + \overline{u_i' u_j'} \frac{\partial \overline{U}_k}{\partial x_j} + \overline{u_k' u_j'} \frac{\partial \overline{U}_i}{\partial x_j} + \frac{\overline{u_j' \partial u_i' u_k'}}{\partial x_j} = \delta_{k3} \overline{u_i' \left(\frac{\theta_v'}{\overline{\theta}_v}\right)} g + \delta_{\iota 3} \overline{u_k' \left(\frac{\theta_v'}{\overline{\theta}_v}\right)} g$$

$$+ f_c \varepsilon_{kj3}\, \overline{u_i' u_j'} + f_c \varepsilon_{ij3}\, \overline{u_k' u_j'} - \overline{\left(\frac{u_i}{\overline{\rho}}\right) \frac{\partial p'}{\partial x_k}} - \overline{\left(\frac{u_k'}{\overline{\rho}}\right) \frac{\partial p'}{\partial x_i}} + \nu\, \overline{u_i' \frac{\partial^2 u_k'}{\partial x_j^2}} + \nu\, \overline{u_k' \frac{\partial^2 u_i'}{\partial x_j^2}}$$

The turbulent continuity equation multiplied by $u_i'u_k'$ and averaged $(\overline{u_i'u_k'\,\partial u_j'/\partial x_j} = 0)$ can be added to the last term before the equal sign to put it into flux form.

Each pressure term can be rewritten using the product rule of calculus, as shown in the following example: $\overline{u_k'\,\partial p'/\partial x_i} = \partial(\overline{p'u_k'})/\partial x_i - \overline{p'\,\partial u_k'/\partial x_i}$. Also, the viscosity terms can be rearranged using the product rule (left as an exercise to the reader). The final form for the momentum flux budget equation is:

$$\underset{\text{I}}{\frac{\partial\left(\overline{u_i'u_k'}\right)}{\partial t}} + \underset{\text{II}}{\frac{\overline{U}_j\partial\left(\overline{u_i'u_k'}\right)}{\partial x_j}} = \underset{\text{III}}{-\left(\overline{u_i'u_j'}\right)\frac{\partial \overline{U}_k}{\partial x_j}} \underset{\text{III}}{-\left(\overline{u_k'u_j'}\right)\frac{\partial \overline{U}_i}{\partial x_j}} \underset{\text{IV}}{-\frac{\partial\left(\overline{u_i'u_j'u_k'}\right)}{\partial x_j}}$$

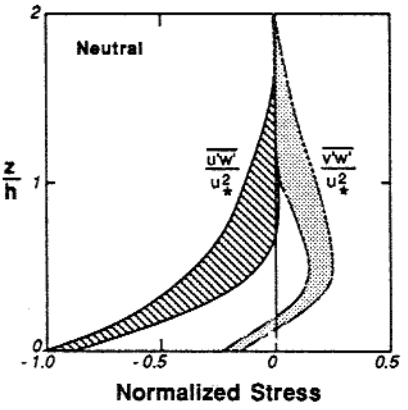
$$+\left(\frac{g}{\overline{\theta}_v}\right)\left[\underset{\text{V}}{\delta_{k3}\,\overline{u_i'\theta_v'}} + \underset{\text{V}}{\delta_{i3}\,\overline{u_k'\theta_v'}}\right] + f_c\left[\underset{\text{VI}}{\varepsilon_{kj3}\,\overline{u_i'u_j'}} + \underset{\text{VI}}{\varepsilon_{ij3}\,\overline{u_k'u_j'}}\right]$$

$$-\frac{1}{\overline{\rho}}\left[\underset{\text{VII}}{\frac{\partial\left(\overline{p'u_k'}\right)}{\partial x_i}} + \underset{\text{VII}}{\frac{\partial\left(\overline{p'u_i'}\right)}{\partial x_k}} - \underset{\text{VIII}}{\overline{p'\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)}}\right] + \underset{\text{IX}}{\frac{\nu\partial^2\left(\overline{u_i'u_k'}\right)}{\partial x_j^2}} - \underset{\text{X}}{\frac{2\nu\overline{\partial u_i'\partial u_k'}}{\partial x_j^{\,2}}} \tag{4.4.1a}$$

Term I	Storage of momentum flux $\overline{u_i'u_k'}$.
Term II	Advection of momentum flux by the mean wind.
Terms III	Production of momentum flux by the mean wind shears.
Term IV	Transport of momentum flux by turbulent motions (turbulent diffusion)
Term V	Buoyant production or consumption.
Term VI	Coriolis effects.
Term VII	Transport by the pressure correlation term (pressure diffusion).
Term VIII	Redistribution by the return-to-isotropy term (named by analogy with the velocity variance equation).
Term IX	Molecular diffusion of turbulent momentum flux.
Term X	Viscous dissipation term (named by analogy with the velocity variance equation). This term is often abbreviated by the symbol $2\varepsilon_{u_iu_k}$.

Scaling arguments, based on observations of the magnitudes of the terms in the above equation, suggest that terms VI, VII, and IX are smaller than the rest, leaving:

Fig. 4.14
Normalized stress vs. height in statically neutral conditions, where h (≈2km) is the height where $\overline{V}$ is zero. Based on large eddy simulation by Mason and Thomson (1987) using U_g=10m/s, V_g=0, and u_*= 0.4 m/s.



$$\frac{\partial\left(\overline{u_i'u_k'}\right)}{\partial t} + \frac{\overline{U}_j \partial\left(\overline{u_i'u_k'}\right)}{\partial x_j} = -\left(\overline{u_i'u_j'}\right)\frac{\partial \overline{U}_k}{\partial x_j} - \left(\overline{u_k'u_j'}\right)\frac{\partial \overline{U}_i}{\partial x_j} - \frac{\partial\left(\overline{u_i'u_j'u_k'}\right)}{\partial x_j}$$

I II III III IV

$$+ \left(\frac{g}{\overline{\theta}_v}\right)\left[\delta_{k3}\,\overline{u_i'\theta_v'} + \delta_{i3}\,\overline{u_k'\theta_v'}\right] + \overline{\frac{p'}{\overline{\rho}}\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)} - 2\varepsilon_{u_iu_k} \qquad (4.4.1b)$$

V V VIII X

Each term in the equation above contains unrepeated i and k indices. Remembering that i and k can each take on three values, that means (4.4.1a or b) represents 9 separate equations. Thus, the above equations can be used to forecast each of the nine terms in the Reynolds stress tensor, although as stated in Chapt. 2 the number of independent terms is reduced to 6 by symmetries.

As an example of an application of this equation for one term, choose a coordinate system aligned with the mean wind. Neglect subsidence and assume horizontal homogeneity. The $\overline{u'w'}$ component (i=1, k=3) of (4.4.1b) is thus:

$$\frac{\partial\left(\overline{u'w'}\right)}{\partial t} = -\frac{\overline{w'^2}\partial\overline{U}}{\partial z} - \frac{\partial\left(\overline{u'w'w'}\right)}{\partial z} + \frac{g\overline{u'\theta_v'}}{\overline{\theta}_v} + \overline{\frac{p'}{\overline{\rho}}\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)} - 2\varepsilon_{uw}$$

(4.4.1c)

In general, the molecular (viscous) dissipation terms for the variance and covariance (flux) equations are abbreviated as $2\varepsilon_\xi$,where ξ represents the variance or covariance.

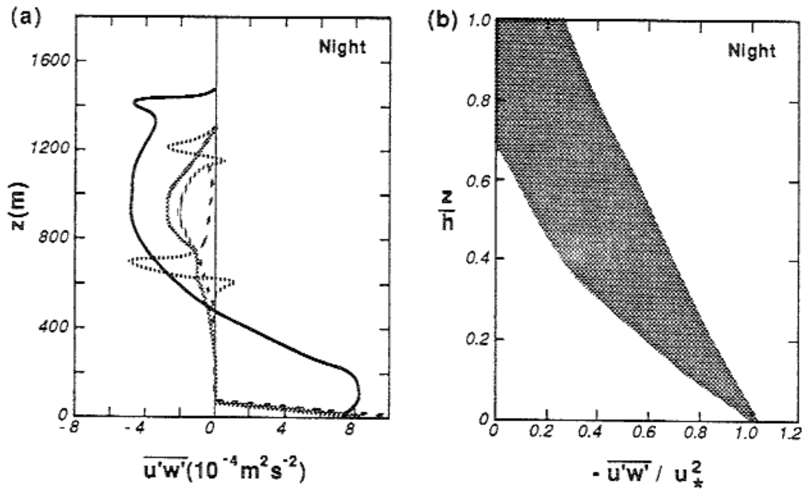


Fig. 4.15 (a) Modeled profiles of turbulent flux of u-wind during Night 33-34, Wangara: symbols same as in Fig. 4.3(a). (b) Range of momentum flux normalized by the friction velocity versus height scaled by boundary layer depth. Based on data from André, et al. (1978), and Caughey et al. (1979).

The only exception is the momentum variance equation (4.3.1g), where 2ε without a subscript is usually used.

Case Study Examples. Fig 4.13 shows vertical profiles of $\overline{u'w'}$ and $\overline{v'w'}$ for daytime cases, while Fig 4.14 shows a neutral example. Nighttime fluxes (Fig 4.15) are often much weaker than daytime fluxes, except near the ground where wind shear at night can maintain the turbulence intensity. The surface values of these fluxes are almost always of the sign appropriate for bringing momentum down from aloft. For mid-latitude situations with predominantly westerly flow that increases speed with height in the surface layer (i.e., $\overline{U}$ is positive), we find that the momentum flux ($\overline{u'w'}$) is negative near the ground. In Fig 4.15a, the surface flux is positive because the mean surface wind in from the east (i.e., $\overline{U}$ is negative).

Fig 4.16 shows the contribution of many of the terms in (4.4.1b) to the overall budgets of $\overline{u'w'}$ (eastward momentum flux budget) and $\overline{v'w'}$ (northward momentum flux budget). Again, steady state is assumed and mean advection is neglected. The resulting values of terms at any one height should thus sum to zero. Large values of many of the terms are observed both at the top and the bottom of the ML, where the strongest mean wind shears are found.

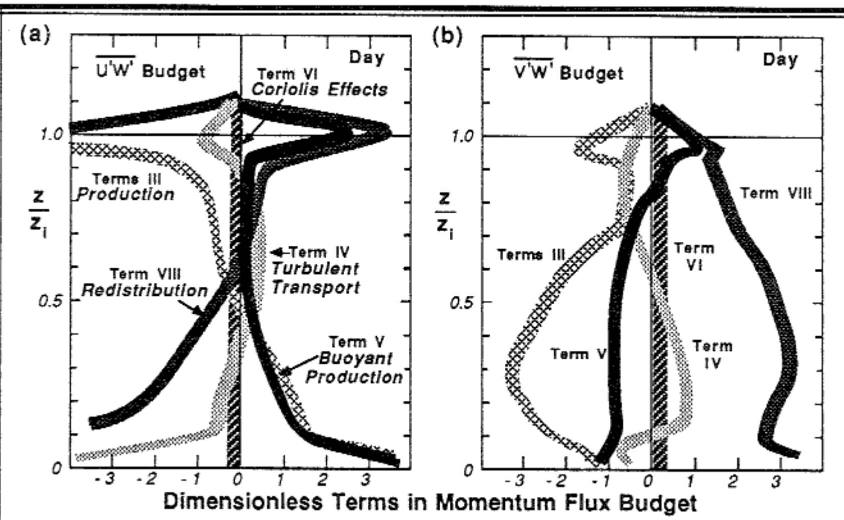


Fig. 4.16 (a) Eastward momentum flux budget as modeled for Voves experiment made dimensionless by dividing by w_*^3 / z_i : (▬), pressure effect; (xxxxx), shear production; (▬), buoyant production; (▒), turbulent transport; (▬), Coriolis effect. (b) Dimensionless northward momentum flux budget as modeled for Voves experiment. After Therry and Lacarrere (1983).

4.4.2 Moisture Flux

Budget Equations. As for momentum flux, the derivation combines two perturbation equations to produce a flux equation. For the first equation start with the momentum perturbation equation (4.1.1), multiply it by the moisture perturbation q', and Reynolds average:

$$\overline{q'\frac{\partial u_i'}{\partial t}} + \overline{U}_j \overline{q'\frac{\partial u_i'}{\partial x_j}} + \overline{q'u_j'}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{q'u_j'\frac{\partial u_i'}{\partial x_j}} =$$

$$+ \overline{q'\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta}_v}\right)} g + f_c\varepsilon_{ij3}\overline{u_j'q'} - \overline{\left(\frac{q'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} + \nu\overline{q'\frac{\partial^2 u_i'}{\partial x_j^2}}$$

Similarly for the second equation, start with the moisture perturbation equation (4.1.2c) and multiply by u_i' and Reynolds average:

$$\overline{u_i'\frac{\partial q'}{\partial t}} + \overline{U}_j\overline{u_i'\frac{\partial q'}{\partial x_j}} + \overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j} + \overline{u_i'u_j'\frac{\partial q'}{\partial x_j}} = \nu_q\overline{u_i'\frac{\partial^2 q'}{\partial x_j^2}}$$

Next, add these two equations, put the turbulent flux divergence term into flux form using the turbulent continuity equation, and combine terms:

$$\frac{\partial\left(\overline{q'u_i'}\right)}{\partial t} + \overline{U}_j \frac{\partial\left(\overline{q'u_i'}\right)}{\partial x_j} + \overline{q'u_j'}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j} + \frac{\partial\left(\overline{q'u_j'u_i'}\right)}{\partial x_j} =$$

$$+ \delta_{i3}\left(\frac{\overline{q'\theta_v'}}{\overline{\theta}_v}\right) g + f_c \varepsilon_{ij3}\,\overline{u_j'q'} - \overline{\left(\frac{q'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} + \nu\,\overline{q'\frac{\partial^2 u_i'}{\partial x_j^2}} + \nu_q\,\overline{u_i'\frac{\partial^2 q'}{\partial x_j^2}}$$

Then, split the pressure term into two parts, and assume $\nu \equiv \nu_q$ to combine the molecular diffusion terms:

$$\underset{\text{I}}{\frac{\partial\left(\overline{q'u_i'}\right)}{\partial t}} + \underset{\text{II}}{\overline{U}_j \frac{\partial\left(\overline{q'u_i'}\right)}{\partial x_j}} = \underset{\text{III}}{-\overline{q'u_j'}\frac{\partial \overline{U}_i}{\partial x_j}} \underset{\text{XI}}{- \overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j}} \underset{\text{IV}}{- \frac{\partial\left(\overline{q'u_j'u_i'}\right)}{\partial x_j}}$$

$$\underset{\text{V}}{+ \delta_{i3}\left(\frac{\overline{q'\theta_v'}}{\overline{\theta}_v}\right) g} \underset{\text{VI}}{+ f_c \varepsilon_{ij3}\left(\overline{u_j'q'}\right)} - \left(\frac{1}{\overline{\rho}}\right)\left[\underset{\text{VII}}{\frac{\partial\left(\overline{p'q'}\right)}{\partial x_i}} \underset{\text{VIII}}{- \overline{p'\frac{\partial q'}{\partial x_i}}}\right]$$

$$\underset{\text{IX}}{+ \frac{\nu\partial^2\left(\overline{q'u_i'}\right)}{\partial x_j^2}} \underset{\text{X}}{- 2\nu\,\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial q'}{\partial x_j}\right)}} \qquad (4.4.2a)$$

The terms in this equation have meanings analogous to those in the momentum flux equation (4.4.1a), except for the additional term (XI), which is a production/loss term related to the mean moisture gradient. Remember that an additional term must be added if the body source is assumed to have perturbations too.

Substituting $2\varepsilon_{u_iq}$ for the last term, and neglecting the Coriolis term, the pressure diffusion term, and the molecular diffusion term leaves:

$$\underset{\text{I}}{\frac{\partial\left(\overline{q'u_i'}\right)}{\partial t}} + \underset{\text{II}}{\overline{U}_j \frac{\partial\left(\overline{q'u_i'}\right)}{\partial x_j}} = \underset{\text{III}}{-\overline{q'u_j'}\frac{\partial \overline{U}_i}{\partial x_j}} \underset{\text{XI}}{- \overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j}} \underset{\text{IV}}{- \frac{\partial\left(\overline{q'u_j'u_i'}\right)}{\partial x_j}}$$

$$+ \delta_{i3}\left(\frac{\overline{q'\theta_v'}}{\overline{\theta}_v}\right) g + \left(\frac{1}{\overline{\rho}}\right)\left[\overline{\frac{p'\partial q'}{\partial x_i}}\right] - 2\,\varepsilon_{u_i q} \qquad (4.4.2b)$$

V VIII X

Terms I and II are the storage and advection, terms III, XI, and V relate to production/consumption; term IV is turbulent transport; term VIII is redistribution; and term X is the molecular destruction (dissipation) of turbulent moisture flux.

Physically, term V relates the correlation (covariance) between moisture and temperature to the production of moisture flux. One would expect that warmer air rises (3.3.3b); thus, if warmer air is also moister (i.e., a positive correlation), then the moist air would probably rise, thereby contributing to the moisture flux.

Physically, term XI suggests production of moisture flux when there is a momentum flux in a mean moisture gradient. The turbulent momentum flux implies a turbulent movement of air. If that movement occurs across a mean moisture gradient, then moisture fluctuation would be expected, as is suggested by analogy to Fig 2.13.

For the special case of vertical moisture flux (i=3) in a horizontally homogeneous setting with no subsidence, (4.4.2b) reduces to:

$$\frac{\partial\left(\overline{q'w'}\right)}{\partial t} = -\overline{w'^2}\,\frac{\partial\overline{q}}{\partial z} - \frac{\partial\left(\overline{q'w'w'}\right)}{\partial z} + \left(\overline{q'\theta_v'}\right)\frac{g}{\overline{\theta}_v} + \left(\frac{1}{\overline{\rho}}\right)\left[\overline{p'\frac{\partial q'}{\partial z}}\right] - 2\,\varepsilon_{wq}$$

I XI IV V VIII X

Case Study Examples. The slope of the moisture flux curve in Fig 4.17 indicates that dry air is being entrained at the top of the ML fast enough to reduce the mean ML

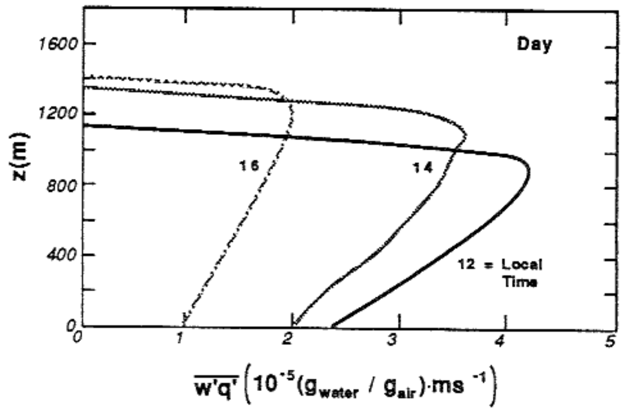


Fig. 4.17 Modeled profiles of turbulent humidity flux during Day 33, Wangara. After André et al., (1978).

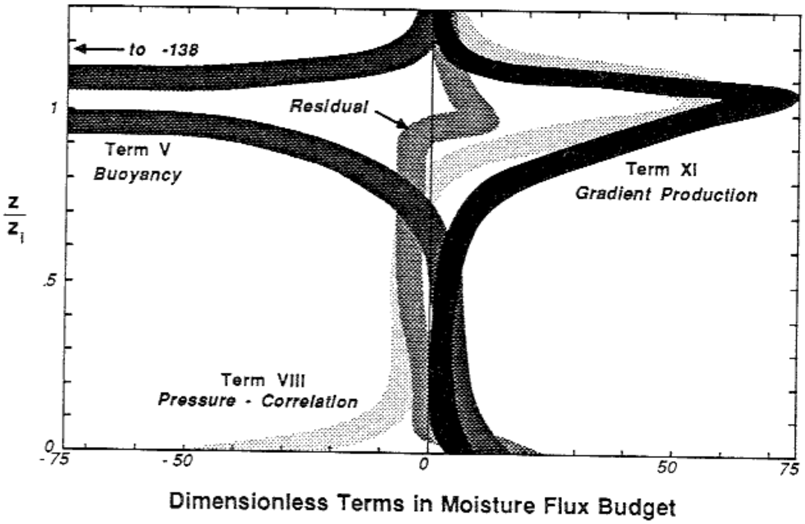


Fig. 4.18 Range of vertical profiles of terms in the moisture flux budget equation for Day 33 Wangara, made dimensionless by dividing by $w_*^2 q_*^{ML}/z_i$. Based on data from Deardorff (1974) and Lenschow, et al. (1980).

humidity, in spite of evaporation from the surface. Fig 4.18 shows a moisture flux budget, with the greatest production and loss values at the top of the ML for this case. Measurements of some of the individual terms are also shown in that figure.

4.4.3 Heat Flux

Budget Equations. The heat flux derivation is similar to that of the moisture flux.

$$\underset{\text{I}}{\frac{\partial\left(\overline{\theta' u_i'}\right)}{\partial t}} + \underset{\text{II}}{\overline{U}_j \frac{\partial\left(\overline{\theta' u_i'}\right)}{\partial x_j}} = \underset{\text{III}}{-\overline{\theta' u_j'}\frac{\partial \overline{U}_i}{\partial x_j}} \underset{\text{XI}}{- \overline{u_i' u_j'}\frac{\partial \overline{\theta}}{\partial x_j}} \underset{\text{IV}}{- \frac{\partial\left(\overline{\theta' u_j' u_i'}\right)}{\partial x_j}}$$

$$\underset{\text{V}}{+ \delta_{i3}\left(\frac{\overline{\theta'\theta_v'}}{\overline{\theta}_v}\right) g} \underset{\text{VI}}{+ f_c \varepsilon_{ij3}\left(\overline{u_j'\theta'}\right)} - \left(\frac{1}{\overline{\rho}}\right)\left[\underset{\text{VII}}{\frac{\partial\left(\overline{p'\theta'}\right)}{\partial x_i}} \underset{\text{VIII}}{- \overline{p'\frac{\partial \theta'}{\partial x_i}}}\right]$$

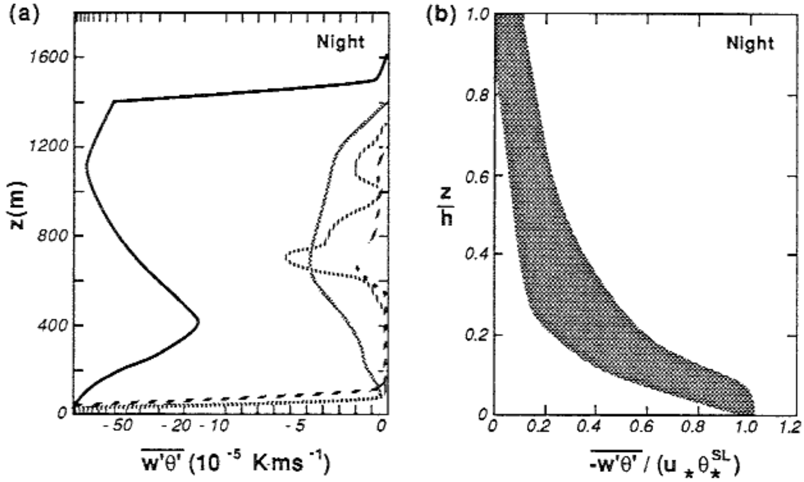


Fig. 4.19 (a) Modeled profiles of kinematic heat flux during Night 33-34, Wangara: symbols as in Fig. 4.3(a); abscissa changes from linear to logarithmic at -10. (b) Range of heat flux normalized by its surface value versus height normalized by boundary layer depth. Based on data from André, et al. (1978) and Caughey et al. (1979).

$$+ \underbrace{\frac{\nu \partial^2 \left(\overline{\theta' u_i'}\right)}{\partial x_j^2}}_{\text{IX}} - \underbrace{2\nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial \theta'}{\partial x_j}\right)}}_{\text{X}} - \underbrace{\left(\frac{1}{\bar{\rho}\, C_p}\right) \overline{u_i' \frac{\partial Q_j^{*\prime}}{\partial x_j}}}_{\text{XII}} \tag{4.4.3a}$$

Namely, we started with the velocity perturbation equation multiplied by θ', and the temperature perturbation equation multiplied by u_i'. Both equations were Reynolds averaged and summed. The turbulent continuity equation was used to put the turbulence diffusion term into flux form.

The terms in this equation have analogous meanings as for the moisture flux equation (4.4.2a). Often, term V is approximated by $\delta_{i3}\, g\, (\overline{\theta_v'^2}/\overline{\theta_v})$. Term XII describes the correlation between velocity fluctuations and with radiation fluctuations.

Substituting $2\, \varepsilon_{u_i\theta}$ for term X, and neglecting the Coriolis, pressure diffusion, radiation, and the molecular diffusion terms for simplicity leaves:

$$\frac{\partial\left(\overline{\theta' u_i'}\right)}{\partial t} + \overline{U}_j \frac{\partial\left(\overline{\theta' u_i'}\right)}{\partial x_j} = -\overline{\theta' u_j'}\frac{\partial \overline{U}_i}{\partial x_j} - \overline{u_i' u_j'}\frac{\partial \overline{\theta}}{\partial x_j} - \frac{\partial\left(\overline{\theta' u_j' u_i'}\right)}{\partial x_j}$$

I II III XI IV

$$+ \delta_{i3}\left(\frac{\overline{\theta'\theta_v'}}{\overline{\theta}_v}\right) g + \left(\frac{1}{\overline{\rho}}\right)\left[\overline{\frac{p'\partial\theta'}{\partial x_i}}\right] - 2\,\varepsilon_{u_i\theta} \qquad (4.4.3b)$$

V VIII X

For the special case of vertical heat flux (i=3) in a horizontally homogeneous setting with no subsidence, (4.4.3b) reduces to:

$$\frac{\partial\left(\overline{\theta' w'}\right)}{\partial t} = -\overline{w'^2}\frac{\partial \overline{\theta}}{\partial z} - \frac{\partial\left(\overline{\theta' w' w'}\right)}{\partial z} + \left(\overline{\theta'\theta_v'}\right)\left(\frac{g}{\overline{\theta}_v}\right) + \left(\frac{1}{\overline{\rho}}\right)\left[\overline{p'\frac{\partial\theta'}{\partial z}}\right] - 2\,\varepsilon_{w\theta}$$

I XI IV V VIII X

(4.4.3c)

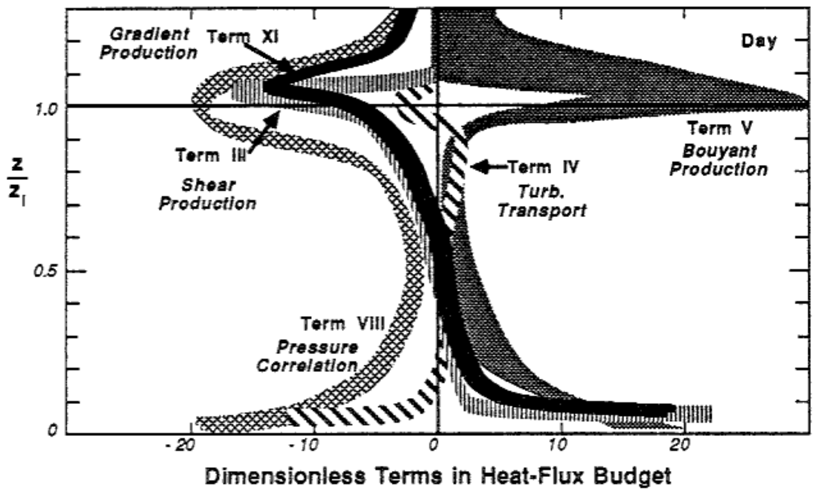


Fig. 4.20 Range of vertical profiles of terms in the heat flux budget equation for Day 33 Wangara, made dimensionless by dividing by $w_*^2\theta_*^{ML}/z_i$. Based on data from Deardorff (1974), Lenschow et al. (1980), André et al. (1978), Therry and Lacarrere (1983) and Zhou et al. (1985).

Case Study Examples. The linear decrease of heat flux with height has been discussed in chapter 3 for the daytime cases (Figs 3.1-3.3 and 3.7). Nighttime heat flux values are much weaker, as shown in Fig 4.19.

Fig 4.20 shows the steady-state, non-advective heat flux budget terms. Large values of terms occur at the surface because of the strong turbulence and mean temperature gradients there. In the middle of the ML, weak potential temperature gradients are associated with smaller magnitudes of the terms there. At the top of the ML the terms become large again, associated with the strong temperature contrast across the ML top.

4.4.4 Flux of a Scalar (Pollutant or Tracer Flux)

Using the same procedures as before, the pollutant flux budget equation is

$$\underset{\text{I}}{\frac{\partial\left(\overline{c'u_i'}\right)}{\partial t}} + \underset{\text{II}}{\overline{U}_j \frac{\partial\left(\overline{c'u_i'}\right)}{\partial x_j}} = - \underset{\text{III}}{\overline{c'u_j'} \frac{\partial \overline{U}_i}{\partial x_j}} - \underset{\text{XI}}{\overline{u_i'u_j'} \frac{\partial \overline{C}}{\partial x_j}} - \underset{\text{IV}}{\frac{\partial\left(\overline{c'u_j'u_i'}\right)}{\partial x_j}}$$

$$\underset{\text{V}}{+\delta_{i3}\left(\frac{\overline{c'\theta_v'}}{\overline{\theta}_v}\right)g} + \underset{\text{VI}}{f_c\varepsilon_{ij3}\left(\overline{u_j'c'}\right)} - \left(\frac{1}{\overline{\rho}}\right)\left[\underset{\text{VII}}{\frac{\partial\left(\overline{p'c'}\right)}{\partial x_i}} - \underset{\text{VII}}{\frac{\overline{p'\partial c'}}{\partial x_i}}\right] + \underset{\text{IX}}{\frac{\nu\partial^2\left(\overline{c'u_i'}\right)}{\partial x_j^2}} \underset{\text{X}}{- 2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial c'}{\partial x_j}\right)}} \qquad (4.4.4a)$$

The terms in this equation have meanings analogous to those of the moisture flux equation (4.4.2a). Substituting 2 ε_{u_ic} for the last term, and neglecting the Coriolis term, the pressure diffusion term, and the molecular diffusion term leaves:

$$\underset{\text{I}}{\frac{\partial\left(\overline{c'u_i'}\right)}{\partial t}} + \underset{\text{II}}{\overline{U}_j \frac{\partial\left(\overline{c'u_i'}\right)}{\partial x_j}} = - \underset{\text{III}}{\overline{c'u_j'} \frac{\partial \overline{U}_i}{\partial x_j}} - \underset{\text{XI}}{\overline{u_i'u_j'} \frac{\partial \overline{C}}{\partial x_j}} - \underset{\text{IV}}{\frac{\partial\left(\overline{c'u_j'u_i'}\right)}{\partial x_j}}$$

$$+ \underset{\text{V}}{\delta_{i3}\left(\frac{\overline{c'\theta_v'}}{\overline{\theta}_v}\right)g} + \underset{\text{VIII}}{\left(\frac{1}{\overline{\rho}}\right)\left[\overline{p'\frac{\partial c'}{\partial x_i}}\right]} \underset{\text{X}}{- 2\,\varepsilon_{u_ic}} \qquad (4.4.4b)$$

For the special case of vertical pollutant flux (i=3) in a horizontally homogeneous setting with no subsidence, (4.4.4b) reduces to:

$$\frac{\partial\left(\overline{c'w'}\right)}{\partial t} = -\overline{w'^2}\frac{\partial \overline{C}}{\partial z} - \frac{\partial\left(\overline{c'w'w'}\right)}{\partial z} + \overline{c'\theta_v'}\left(\frac{g}{\overline{\theta}_v}\right) + \left(\frac{1}{\overline{\rho}}\right)\left[\overline{p'\frac{\partial c'}{\partial z}}\right] - 2\,\varepsilon_{wc} \qquad (4.4.4c)$$

4.4.5 Buoyancy Flux

Both the definition of w_* and the buoyant production term in equation (4.3.1j) contain a buoyancy flux defined by $(g/\overline{\theta}_v)\ (\overline{w'\theta_v'})_s$. The flux of virtual potential temperature $\overline{w'\theta_v'}$ is different than the heat flux $\overline{w'\theta'}$ — the two must not be interchanged. We can, however, use the definition of virtual potential temperature to derive a diagnostic equation for $\overline{w'\theta_v'}$ in terms of $\overline{w'\theta'}$.

Start with

$$\theta_v \cong \theta\ \left[\ 1\ +\ 0.61r\ -\ r_L\right] \qquad (4.4.5a)$$

from section 1.5 (or Appendix D), where it is understood that r equals the saturation value whenever r_L is nonzero. Expand the dependent variables into mean and turbulent parts:

$$\overline{\theta}_v + \theta_v' = \left(\overline{\theta} + \theta'\right)\left[\ 1\ +\ 0.61\left(\overline{r} + r'\right) - \left(\overline{r_L} + r_L'\right)\right]$$

$$= \overline{\theta}\left[1 + 0.61\overline{r} - \overline{r_L}\right] + \overline{\theta}\left[\ 0.61r' - r_L'\right] + \theta'\left[1 + 0.61\overline{r} - \overline{r_L}\right] + \theta'\left[0.61r' - r_L'\right]$$

Multiply this equation by w' and Reynolds average

$$\overline{w'\theta_v'} = \overline{\theta}\left[0.61\left(\overline{w'r'}\right) - \left(\overline{w'r_L'}\right)\right] + \left(\overline{w'\theta'}\right)\left[1 + 0.61\overline{r} - \overline{r_L}\right] + \overline{w'\theta'\left[0.61r' - r_L'\right]} \qquad (4.4.5b)$$

The last terms are triple correlations ($\overline{w'\theta'r'}$ and $\overline{w'\theta'r_L'}$). Observations in the atmosphere suggest that they are small enough compared to the other terms to be neglected, although these estimates are difficult to measure and fraught with error. Thus, the usual form for the virtual heat flux is:

$$\overline{w'\theta_v'} \cong \overline{\theta}\left[0.61\left(\overline{w'r'}\right) - \left(\overline{w'r_L'}\right)\right] + \left(\overline{w'\theta'}\right)\left[1 + 0.61\overline{r} - \overline{r_L}\right] \qquad (4.4.5c)$$

For the special case of no liquid water in the air, this reduces to:

$$\overline{w'\theta_v'} \cong \left(\overline{w'\theta'}\right)\left[\,1 + 0.61\bar{r}\,\right] + 0.61\bar{\theta}\left(\overline{w'r'}\right) \qquad (4.4.5d)$$

Modelers and theorists often have no recourse but to use this equation whenever buoyancy flux is needed. Experimentalists, on the other hand, often have direct observations of instantaneous values of θ and r, enabling them to calculate θ_v using (4.4.5a). Knowing θ_v, it is easy to then find $\overline{\theta_v}$ and θ_v', as is done with any other variable. The resulting flux found by forming the product of w' and θ_v' yields a more accurate virtual potential temperature flux than (4.4.5c or d).

Case study examples for the buoyancy flux were shown in Figs 3.1-3.3.

4.5 References

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4.6 Exercises

1) Confirm that $-2\,\overline{u_j'q'}\,\partial\overline{q}/\partial x_j$ is a production term and not a loss term for equation (4.3.2). Hint, review section 2.7 and Fig 2.13.
2) Given values for the viscous dissipation rate of velocity variance (see section 4.3.1), express that rate as a heating rate $\partial\overline{\theta}/\partial t$ for air, and compare its magnitude with the magnitudes of other terms in (3.4.5b). Hint, remember that viscosity dissipates turbulent motions into heat.
3) Given the general form for the momentum flux equation (4.4.1b), write out the equations for the following components:

   a) $\overline{u'w'}$ b) $\overline{v'w'}$ c) $\overline{u'v'}$
4) Given the momentum flux equation (4.4.1b), show how to transform that equation into an equation for velocity variance $\overline{u_i'^2}$.
5) Show how the two viscosity terms in the equation just before (4.4.1a) can be manipulated into the form shown in (4.4.1a). Hint, start with $\partial^2(\overline{u_i'u_k'})/\partial x_j^2$.
6) a) Given the data from Figs 3.1-3.6 of chapter 3, calculate w_* for each of the flights.

   b) Also calculate t_*, θ_*^{ML}, and q_*^{ML}.
7) In Fig 3.1a of chapter 3 are plotted two data points at each height. One data point represents heat flux and one represents moisture flux. Using the values from this figure, calculate $\overline{w'\theta_v'}$ for each of those heights, and plot the result. Do NOT normalize your results by the surface value.
8) Given Fig 4.13, calculate the value of terms VI and IX of equation (4.4.1a). By comparing these values with the magnitudes of the other terms, are we justified in neglecting them to derive (4.4.1b)?
9) Some of the terms in (4.4.1a) involve correlations with pressure perturbation. Discuss how you would design an instrument for measuring p' , and what some of the difficulties might be.
10) Many of the prognostic equations in this chapter include triple-correlation terms such as $\overline{w'w'\theta'}$. Discuss the steps (but do not do the complete derivation) you would take to derive a prognostic equation for $\partial(\overline{w'w'\theta'})/\partial t$. Hint, review the general methodology used to derive equations for $\partial(\overline{w'\theta'})/\partial t$.

11) Suppose that your best friend sneezes into still air, creating a SEG cloud of concentration c, where SEG = Someone Else's Germs. These germs either multiply or die depending on the temperature. If the following conservation equation describes SEG: $dc/dt = b\,c\,(T-T_o)$, where b and T_o are constants and T is temperature, then derive a prognostic equation for $\overline{w'c'}$.